

GEOMETRY OF GENERIC MOISHEZON TWISTOR SPACES ON $4\mathbb{CP}^2$

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ABSTRACT. In this paper we investigate a family of Moishezon twistor spaces on the connected sum of 4 complex projective planes, which can be regarded as a direct generalization of the twistor spaces on $3\mathbb{CP}^2$ of double solid type studied by Poon and Kreussler-Kurke. These twistor spaces have a natural structure of double covering over a scroll of 2-planes over a conic. We determine the defining equations of the branch divisors in an explicit form, which are very similar to the case of $3\mathbb{CP}^2$. Using these explicit description we compute the dimension of the moduli spaces of these twistor spaces. Also we observe that similarly to the case of $3\mathbb{CP}^2$, these twistor spaces can also be considered as generic Moishezon twistor spaces on $4\mathbb{CP}^2$. We obtain these results by analyzing the anticanonical map of the twistor spaces in detail, which enables us to give an explicit construction of the twistor spaces, up to small resolutions.

1. INTRODUCTION

In their papers, Kreussler-Kurke [12] and Poon [16] investigated algebraic structure of generic twistor spaces on $3\mathbb{CP}^3$, the connected sum of 3 copies of complex projective planes. They showed that if the half-anticanonical system of a twistor space of $3\mathbb{CP}^2$ is base point free, then the morphism associated to the system becomes a generically 2 to 1 covering map whose branch divisor is a quartic surface. Further, they determined defining equation of the quartic surface; for the most generic twistor spaces, with respect to homogeneous coordinates on \mathbb{CP}^3 , the equation is of the form

$$(1.1) \quad z_0 z_1 z_2 z_3 = Q(z_0, z_1, z_2, z_3)^2$$

where Q is a (homogeneous) quadratic polynomial with real coefficients. From the equation, the intersection of the quadratic surface $Q = 0$ and any of the 4 plane $z_i = 0$ is a double conic, and the intersection points of these 4 conics (consisting of 12 points) are ordinary double points of the quartic surface (1.1). For a generic quadratic polynomial Q , these are all singularities of the surface (1.1), but they showed that when (1.1) is actually the branch divisor of the twistor spaces, the surface has one more node, which is necessarily real, so that the branch surface has 13 ordinary nodes in total.

In this paper, we shall find Moishezon twistor spaces on $4\mathbb{CP}^2$ which can be regarded as a direct generalization of these twistor spaces on $3\mathbb{CP}^2$. More concretely, we show the following. There exist twistor spaces on $4\mathbb{CP}^2$ such that (i) the anticanonical system is 4-dimensional as a linear system, and the image of the associated rational map is a scroll Y of 2-planes in \mathbb{CP}^4 over a conic, (ii) there is an explicit and simple elimination of the indeterminacy locus of the anticanonical map, whose resulting morphisms is a generically 2 to 1 covering map onto the scroll Y , (iii) the branch divisor of the last covering, which will be denoted by B throughout this paper, is an intersection of Y with a quartic hypersurface in \mathbb{CP}^4 , (iv) if we take homogeneous coordinates on \mathbb{CP}^4 such that the scroll Y is defined by $z_0^2 = z_1 z_2$, then the quartic hypersurface is defined by the equation

$$(1.2) \quad z_0 z_3 z_4 f(z_0, z_1, z_2, z_3, z_4) = Q(z_0, z_1, z_2, z_3, z_4)^2,$$

where f and Q are linear and quadratic polynomials with real coefficients respectively. The double covering structure and similarity of the equations (1.2) with (1.1) would justify to call these twistor spaces a direct generalization of those by Kreussler-Kurke-Poon.

The main tool of the present investigation is the anticanonical system of the twistor spaces. In Section 2 we start by constructing a rational surface S which will be contained in the twistor space Z on $4\mathbb{CP}^2$ as a real half-anticanonical divisor, and then clarify the structure of bi-anticanonical system on S . Next in Section 3 we study the structure of the anticanonical map of the twistor spaces in detail. In Section 3.1 we show that the anticanonical map induces a rational map to \mathbb{CP}^4 , whose image is a scroll Y of planes over a conic, and give an explicit elimination of the indeterminacy locus, obtaining a degree 2 morphism $Z_1 \rightarrow Y$. Next in Section 3.2 we analyze the structure of the anticanonical map more in detail, and by applying some explicit blowups and blowdowns we modify the degree 2 morphism $Z_1 \rightarrow Y$ to another morphism $Z_4 \rightarrow Y$ so as to have no divisor to be contracted. Up to contraction of curves, this gives the Stein factorization of the morphism $Z_1 \rightarrow Y$. Although the modifications therein are a little bit complicated, they are rather natural in light of the structure of the anticanonical system, and indispensable for obtaining explicit construction of the twistor spaces. We also obtain a key technical result that the branch divisor of $Z_1 \rightarrow Y$ is a cut of Y by a quartic hypersurface.

In Section 4, we explicitly determine a defining equation of the branch divisor of the double covering. For this, in Section 4.1 we find 5 hyperplanes in \mathbb{CP}^4 such that the intersection of the branch divisor with the hyperplanes becomes double curves, i.e. a curve of multiplicity 2. These double curves are analogous to the above 4 conics appeared in the case of $3\mathbb{CP}^2$, but to understand how they intersect each other requires some effort. In addition, in the present case, finding all the double curves is not so easy, since not all of the double curves are obtained as an image of twistor lines as in the case of $3\mathbb{CP}^2$. In Section 4.2 we show that the 5 double curves are contained in a quadratic hypersurface in \mathbb{CP}^4 , and that such a hyperquadric is unique up to the defining equation of the scroll Y . In Section 4.3 we prove the main result which determines the defining equation of the quartic hypersurface (Theorem 4.5). The equation includes not only the quadratic polynomial obtained in Section 4.2 but also a linear polynomial, which might look strange at first sight. We give an account for a geometric meaning of it.

In Section 4.4 we investigate singularities of the branch divisor B of the double covering. As above B has 5 double curves, and at most of the intersection points of them, B has (non-real) ordinary double points. This is totally parallel to the case of $3\mathbb{CP}^2$ explained at the beginning. But in the present case there are exactly 2 special intersection points, at which B has A_3 -singularities. Besides these ordinary double points and A_3 -singular points, we show that B has other isolated singularities and determine their basic invariants (Theorem 4.10). The result means that in general B has extra 6 ordinary double points. For obtaining this result, as in Kreussler [10] and Kreussler-Kurke [12] in the case of $3\mathbb{CP}^2$, we compute the Euler numbers of the relevant spaces, especially the branch divisor B . We also note that the concrete modification of the anticanonical map obtained in Section 3.2 is crucial for determining the invariants of the singularities.

In Section 5.1 we compute dimension of the moduli space of the present twistor spaces. We first compute the dimension by counting the number of effective parameters (coefficients) involved in the defining quartic polynomials. Next we show that some cohomology group of the twistor space can be regarded as a tangent space of the moduli space, and see that it coincides with the dimension obtained by counting the number of effective parameters. This

implies a completeness of our description obtained in Section 4. In Section 5.2 we discuss genericity of our twistor spaces among all Moishezon twistor spaces on $4\mathbb{CP}^2$, indicated in the title of this paper. The genericity implies a kind of density in the moduli space of Moishezon twistor spaces on $4\mathbb{CP}^2$. In the course we also give a rough classification of Moishezon twistor spaces on $4\mathbb{CP}^2$ under some genericity assumption.

In Appendix we show that an inverse of a non-standard contraction map employed in Section 3.2 can be realized by an embedded blowup with a non-singular center in the ambient space, and point out in a concrete form that the modification can be regarded as a singular version of the well-known operation of Hironaka [5] for constructing non-projective Moishezon threefolds.

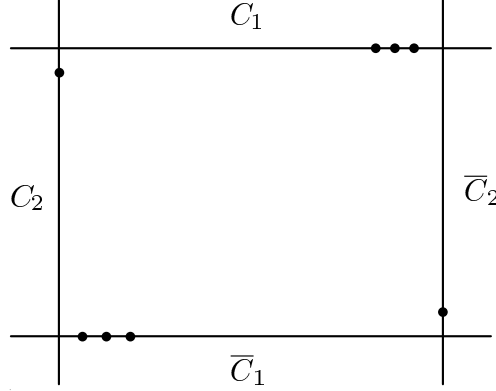
We should also mention a relationship between this work and our previous paper [8]. As we explained in the beginning, in the case of $3\mathbb{CP}^2$, the branch quartic divisor of the double covering becomes of the form (1.1) under a genericity assumption. In non-generic cases, as showed by Kreussler-Kurke [12], the branch divisor becomes similar but more degenerate form, and in the most degenerate situation, the branch divisor has a \mathbb{C}^* -action. As a consequence, in that case the twistor spaces have a \mathbb{C}^* -action. The twistor spaces studied in [8] is a generalization of these twistor spaces (with \mathbb{C}^* -action) to the case of $n\mathbb{CP}^2$, $n > 3$. In this respect we remark that the twistor spaces in this paper is obtained as a deformation of the twistor spaces studied in [8] in the case of $4\mathbb{CP}^2$. It is very natural to expect that we can obtain a further generalization to the case of $n\mathbb{CP}^2$, $n > 4$. We hope to discuss this attractive topic in a future paper.

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Notations and Conventions. The natural square root of the anticanonical bundle of a twistor space is denoted by F . If two varieties X_1 and X_2 are birational under some blowup or blowdown, and if Y is a subvariety of X_1 , then we often use the same symbol Y to mean the strict transform or birational image of Y under the blowup or blowdown, as far as it makes sense. For a line bundle L and a non-zero section s of L , (s) means the zero divisor of s . For a linear subspace $V \subset H^0(L)$, we denote by $|V|$ to mean the linear system $\{(s) \mid s \in H^0(L), s \neq 0\}$. $\text{Bs } |V|$ means the base locus of $|V|$. We mean $\dim |V| = \dim V - 1$ and $h^i(L) = \dim H^i(L)$.

2. A CONSTRUCTION OF RATIONAL SURFACES AND THEIR BI-ANTICANONICAL SYSTEM

We are going to investigate twistor spaces on $4\mathbb{CP}^2$ which contain a particular type of non-singular rational surface S as a real member of $|F|$. In this section we first construct the surface S as a blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1$, and then study the bi-anticanonical system on it. For this, we define the line bundles $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as the pullback of $\mathcal{O}_{\mathbb{CP}^1}(1)$ by the projection to the first and second factors respectively. We simply call members of the linear system $|\mathcal{O}(m,n)|$ as (m,n) -curves. We define a real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a product of the complex conjugation and the antipodal map. Next take any non-real $(1,0)$ -curves C_1 , any $(0,1)$ -curve C_2 , any distinct 3 points on $C_1 \setminus (C_2 \cup \overline{C_2})$, and 1 point on $C_2 \setminus (C_1 \cup \overline{C_1})$. By taking the images under the real structure, we obtain distinct 8 points in total (see Figure 1). Let $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ be the blowup at these 8 points. This surface S has a natural real structure induced from that on $\mathbb{CP}^1 \times \mathbb{CP}^1$, and also has

FIGURE 1. The 8 points to be blown up, giving the surface S .

an anticanonical curve

$$C := C_1 + C_2 + \bar{C}_1 + \bar{C}_2,$$

where this time C_i means the strict transform of the original C_i , so that $C_1^2 = \bar{C}_1^2 = -3$ and $C_2^2 = \bar{C}_2^2 = -1$. The identity component of the holomorphic automorphism group of S is trivial. In the sequel we investigate the anticanonical and bi-anticanonical systems on S .

Proposition 2.1. (i) $\dim |K_S^{-1}| = 0$, so that C is the unique anticanonical curve on S , (ii) $\dim |2K_S^{-1}| = 2$, $\text{Bs} |2K_S^{-1}| = C_1 \cup \bar{C}_1$, and $\text{Bs} |2K_S^{-1} - C_1 - \bar{C}_1| = \emptyset$.

Proof. (i) is immediate. For (ii), $CC_1 = C\bar{C}_1 = -1$ means the inclusion $C_1 \cup \bar{C}_1 \subset \text{Bs} |2K_S^{-1}|$. Riemann-Roch formula and the rationality of S imply $\chi(K_S^{-1}) = 1$. These mean $H^1(K_S^{-1}) = 0$. Therefore restricting $2K_S^{-1} - C_1 - \bar{C}_1$ to $C_2 \cup \bar{C}_2$, we obtain the exact sequence

$$(2.1) \quad 0 \longrightarrow H^0(K_S^{-1}) \longrightarrow H^0(2K_S^{-1} - C_1 - \bar{C}_1) \longrightarrow H^0(\mathcal{O}_{C_2}) \oplus H^0(\mathcal{O}_{\bar{C}_2}) \longrightarrow 0.$$

This implies $(C_2 \cup \bar{C}_2) \cap \text{Bs} |2K_S^{-1} - C_1 - \bar{C}_1| = \emptyset$, and also $h^0(2K_S^{-1} - C_1 - \bar{C}_1) = 3$. On the other hand by restricting the same system to $C_1 \cup \bar{C}_1$, we obtain the exact sequence

$$(2.2) \quad 0 \longrightarrow H^0(\mathcal{O}_S(2C_2 + 2\bar{C}_2)) \longrightarrow H^0(2K_S^{-1} - C_1 - \bar{C}_1) \xrightarrow{r} H^0(\mathcal{O}_{C_1}(1)) \oplus H^0(\mathcal{O}_{\bar{C}_1}(1)).$$

As $h^0(\mathcal{O}_S(2C_2 + 2\bar{C}_2)) = 1$ clearly, the image of the restriction map r in (2.2) is 2-dimensional. Projecting this to $H^0(\mathcal{O}_{C_1}(1))$ gives either a 1-dimensional subspace or $H^0(\mathcal{O}_{C_1}(1))$ itself. In order to prove $\text{Bs} |2K_S^{-1} - C_1 - \bar{C}_1| = \emptyset$, it is enough to exclude the former possibility. Suppose it is the case. Let $p \in C_1$ be the zero point of a generator of the 1-dimensional subspace. Then we have $\text{Bs} |2K_S^{-1} - C_1 - \bar{C}_1| = \{p, \bar{p}\}$ and $p \notin C_2 \cup \bar{C}_2$ as we have already seen. Let $S' \rightarrow S$ be the blowup at p and \bar{p} , and C'_1 and \bar{C}'_1 the strict transforms of C_1 and \bar{C}_1 respectively. As $(2K_S^{-1} - C_1 - \bar{C}_1)^2 = 2$, for any member $D \in |2K_S^{-1} - C_1 - \bar{C}_1|$ with $D \neq C_1 + 2C_2 + \bar{C}_1 + 2\bar{C}_2$, we have $C \cap D = \{p, \bar{p}\}$, and the intersections are transversal. This means that the system $|C'_1 + 2C_2 + \bar{C}'_1 + 2\bar{C}_2|$ on S' is base point free. Then as $C'_1(C'_1 + 2C_2 + \bar{C}'_1 + 2\bar{C}_2) = \bar{C}'_1(C'_1 + 2C_2 + \bar{C}'_1 + 2\bar{C}_2) = 0$, the morphism associated to the system contracts C'_1 and \bar{C}'_1 to points. On the other hand, by (2.1), C_2 and \bar{C}_2 are

mapped to mutually different points by the same morphism. This is a contradiction because $C_1 \cap C_2 \neq \emptyset$, $C_1 \cap \overline{C}_2 \neq \emptyset$, and C_1 is connected. \square

Let $\phi : S \rightarrow \mathbb{CP}^2$ be the morphism associated to the system $|2K_S^{-1}| \simeq |2K_S^{-1} - C_1 - \overline{C}_1|$. Since $\text{Bs } |2K_S^{-1} - C_1 - \overline{C}_1| = \emptyset$ and $(2K_S^{-1} - C_1 - \overline{C}_1)C_2 = (2K_S^{-1} - C_1 - \overline{C}_1)\overline{C}_2 = 0$, ϕ factors as $S \rightarrow \overline{S} \rightarrow \mathbb{CP}^2$, where $S \rightarrow \overline{S}$ denotes the blowdown of C_2 and \overline{C}_2 .

Proposition 2.2. *The morphism ϕ is generically 2 to 1, and the branch divisor is a quartic curve. Further, the images $\phi(C_1)$ and $\phi(\overline{C}_1)$ are the same line, and $\phi(C_2)$ and $\phi(\overline{C}_2)$ are different 2 points on the line.*

Proof. The morphism ϕ is surjective since $(2K_S^{-1} - C_1 - \overline{C}_1)^2 = 2 > 0$. This also means it is generically 2 to 1. Further, a general member D of $|2K_S^{-1} - C_1 - \overline{C}_1|$, which is an irreducible non-singular curve by Bertini's theorem, is an elliptic curve because $D(D + K_S) = 0$. Hence the branch curve of ϕ must be of degree 4. For the images $\phi(C_1)$ and $\phi(\overline{C}_1)$, as in the proof of Proposition 2.1, we have the exact sequence (2.2), and the image of r projects isomorphically to $H^0(\mathcal{O}_{C_1}(1))$ and $H^0(\mathcal{O}_{\overline{C}_1}(1))$. Hence $\phi(C_1)$ and $\phi(\overline{C}_1)$ are lines. Further these lines are identical, since they are precisely the 2-dimensional images into the dual space $\mathbb{P}H^0(2K_S^{-1} - C_1 - \overline{C}_1)^*$ by the dual map of r . Also the last claim for $\phi(C_2)$ and $\phi(\overline{C}_2)$ is clear from the exact sequence (2.1), $C_1 \cap C_2 \neq \emptyset$ and $C_1 \cup \overline{C}_2 \neq \emptyset$. \square

The following property of the branch quartic will also be needed later.

Proposition 2.3. *If we denote the line and the pair of points on it by $l := \phi(C_1) = \phi(\overline{C}_1)$ and $p_2 := \phi(C_2)$ and $\overline{p}_2 = \phi(\overline{C}_2)$ respectively, then p_2 and \overline{p}_2 are smooth points of the branch quartic curve, and the curve is tangent to l at these 2 points. (Hence l is a bitangent of the branch curve.)*

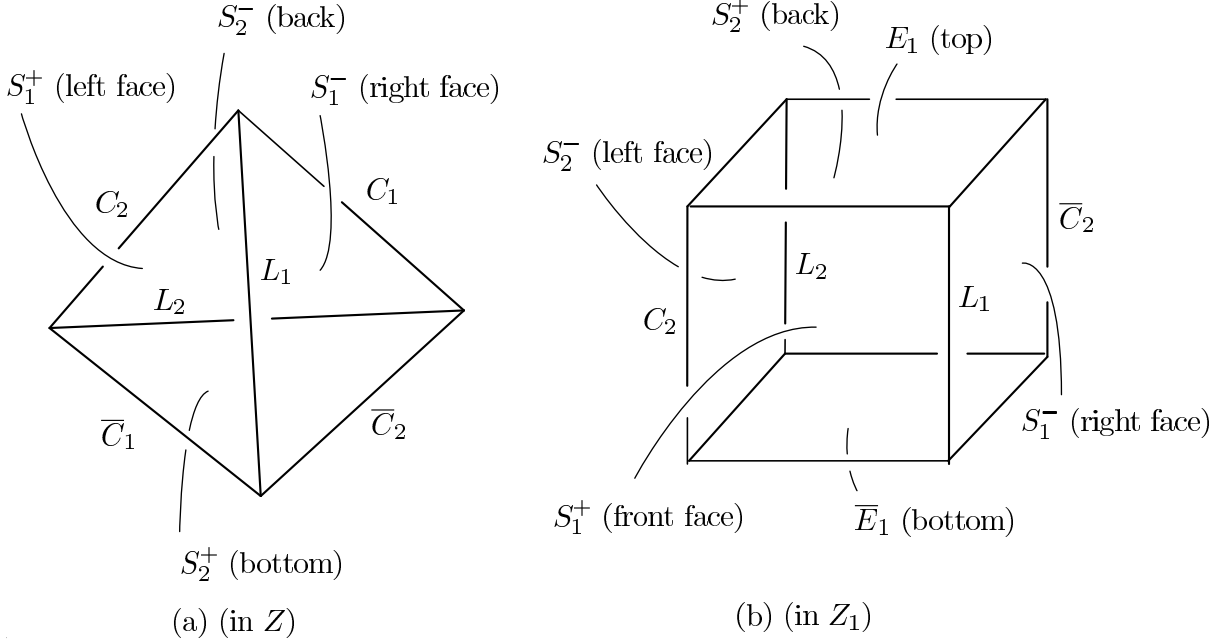
Proof. Let β be the branch curve of ϕ . As $C_1 + 2C_2 + \overline{C}_1 + 2\overline{C}_2 \in |2K_S^{-1} - C_1 - \overline{C}_1|$, there is a line l' such that $\phi^{-1}(l') = C_1 + 2C_2 + \overline{C}_1 + 2\overline{C}_2$. But as $\phi(C_1) = l$, we obtain $l' = l$. Since $\phi(C_2) = p_2$ and both $C_2 \cap C_1$ and $C_2 \cap \overline{C}_1$ are non-empty, it follows that $p_2 \in \beta$, so that $\overline{p}_2 \in \beta$ by the real structure. Further, after the blowdown $S \rightarrow \overline{S}$, the curve $C_1 \cup \overline{C}_1$ is of course locally reducible at the images of C_2 and \overline{C}_2 . Hence we have $\beta|_l = 2p_2 + 2\overline{p}_2$ as divisors. This means that either β has double points at p_2 and \overline{p}_2 , or otherwise β is smooth at p_2 and \overline{p}_2 and is tangent to l at these 2 points. But in the former case by smoothness of S there have to be extra exceptional curves over p_2 and \overline{p}_2 , which contradicts $\phi^{-1}(l) = C_1 + 2C_2 + \overline{C}_1 + 2\overline{C}_2$. Hence the claim follows. \square

3. ANALYSIS OF THE ANTICANONICAL MAP ON THE TWISTOR SPACES

3.1. The anticanonical map of the twistor spaces. Let S be the rational surface equipped with the real structure constructed in the previous section, and $C = C_1 + C_2 + \overline{C}_1 + \overline{C}_2$ the unique anticanonical curve on S . Let Z be a twistor space on $4\mathbb{CP}^2$ and suppose that Z contains S as a real member of $|F|$. The following property of $|F|$ is immediate to see and we omit a proof.

Proposition 3.1. *The system $|F|$ satisfies the following: (i) $\dim |F| = 1$, (ii) $\text{Bs } |F| = C$, (iii) the number of reducible members of $|F|$ is two, and both of the members are real.*

We note that it readily follows from (i) and (ii) that a general member S' of the pencil $|F|$ is also obtained from $\mathbb{CP}^1 \times \mathbb{CP}^1$ by blowing up 8 points arranged as in Figure 1, where the

FIGURE 2. a tetrahedron in Z and a cube in Z_1

positions of the 8 points are not identical to the original ones. We denote the 2 reducible members of $|F|$ by

$$(3.1) \quad S_1 = S_1^+ + S_1^- \quad \text{and} \quad S_2 = S_2^+ + S_2^-,$$

where we make distinction between S_i^+ and S_i^- by declaring that S_1^+ and S_2^+ contains the component \overline{C}_1 . We denote $L_1 := S_1^+ \cap S_1^-$ and $L_2 := S_2^+ \cap S_2^-$, both of which are twistor lines by [16, §1]. Then these divisors and curves form a tetrahedron as illustrated in Figure 2, (a). These will be significant for our analysis of the anticanonical system on the twistor spaces. We show the the following basic properties of the anticanonical system. Note that (iv) means that Z is Moishezon.

Proposition 3.2. *The anticanonical system $|2F| = |K_Z^{-1}|$ of the twistor space Z satisfies the following: (i) $\dim |2F| = 4$, (ii) $\text{Bs } |2F| = C_1 \cup \overline{C}_1$, (iii) if $\mu_1 : Z_1 \rightarrow Z$ denotes the blowup at $C_1 \cup \overline{C}_1$, $E_1 \cup \overline{E}_1$ the exceptional divisor, and $\mathcal{L}_1 := \mu_1^*(2F) - E_1 - \overline{E}_1$, then $\text{Bs } |\mathcal{L}_1| = \emptyset$, (iv) if Φ_1 denotes the morphism associated to $|\mathcal{L}_1|$, then the image $\Phi_1(Z_1)$ is a scroll of 2-planes over a conic, and the morphism Φ_1 is generically 2 to 1 over the scroll.*

By the blowup $\mu_1 : Z_1 \rightarrow Z$, the tetrahedron in Z is transformed to be a cubic in Z_1 as in Figure 2, (b).

Proof of Proposition 3.2. (i) is immediate from Proposition 2.1 (ii), Proposition 3.1 (i), and the exact sequence

$$(3.2) \quad 0 \longrightarrow H^0(F) \longrightarrow H^0(2F) \longrightarrow H^0(2K_S^{-1}) \longrightarrow 0,$$

where the last zero is a consequence of $h^0(F) = 2$ and the Riemann-Roch formula applied to F . The claim (ii) also follows from this exact sequence and Proposition 2.1 (ii).

For (iii), let \tilde{S} be the strict transform of S . Then \tilde{S} is biholomorphic to S , and $\tilde{S} \in |\mu_1^*F - E_1 - \overline{E}_1|$. Hence we have an exact sequence

$$(3.3) \quad 0 \longrightarrow \mu_1^*F \longrightarrow \mu_1^*(2F) - E_1 - \overline{E}_1 \longrightarrow \mu_1^*(2F) - E_1 - \overline{E}_1|_{\tilde{S}} \longrightarrow 0.$$

Since $H^1(\mu_1^*F) \simeq H^1(F) = 0$, we obtain that the restriction map $H^0(\mu_1^*(2F) - E_1 - \overline{E}_1) \rightarrow H^0(\mu_1^*(2F) - E_1 - \overline{E}_1|_{\tilde{S}})$ is surjective. Further, as $(\mu_1^*F)|_{\tilde{S}} \simeq F|_S \simeq K_S^{-1}$ under the biholomorphism $\tilde{S} \simeq S$, we have $\mu_1^*(2F)|_{\tilde{S}} \simeq 2K_S^{-1}$. Further, $E_1|_{\tilde{S}} \simeq \mathcal{O}_S(C_1)$. Hence we obtain an isomorphism $\mu_1^*(2F) - E_1 - \overline{E}_1|_{\tilde{S}} \simeq 2K_S^{-1} - C_1 - \overline{C}_1$. Therefore by the third claim of Proposition 2.1 (ii), we obtain $\text{Bs } |\mu_1^*(2F) - E_1 - \overline{E}_1| = \emptyset$.

Let $\Phi : Z \rightarrow \mathbb{CP}^4$ be the rational map associated to the anticanonical system $|2F|$, so that $\Phi_1 = \Phi \circ \mu_1$. For (iv) it is enough to show that $\Phi(Z)$ is the 3-dimensional scroll as in the statement, and the rational map $\Phi : Z \rightarrow \Phi(Z)$ is generically 2 to 1. Let $S^2H^0(F)$ be the subspace of $H^0(2F)$ generated by all sections of the form s_1s_2 where $s_i \in H^0(F)$. This is a 3-dimensional subspace. Then we have the following left commutative diagram of rational maps:

$$(3.4) \quad \begin{array}{ccc} Z & \xrightarrow{\Phi} & \mathbb{CP}^4 \\ & \searrow f & \downarrow \pi \\ & & \mathbb{CP}^2 \end{array} \quad \begin{array}{ccc} Z & \xrightarrow{\Phi} & Y \\ & \searrow f & \downarrow \pi \\ & & \Lambda \end{array}$$

where π is the linear projection induced by the inclusion $S^2H^0(F) \subset H^0(2F)$ and f is the rational map associated to the subsystem $|S^2H^0(F)|$. Clearly the image $f(Z)$ is a conic, for which we denote by Λ . Hence writing $Y := \pi^{-1}(\Lambda)$, Y is exactly the scroll as in the statement of (iv), and we obtain the right commutative diagram in (3.4). We have to show that $\Phi : Z \rightarrow Y$ is surjective and generically 2 to 1. For these, we note that by the definition of f , for any $\lambda \in \Lambda$, $f^{-1}(\lambda)$ belongs to the pencil $|F|$. Then by (3.2) for *any* non-singular member $S \in |F|$, the restriction $\Phi|_S$ is exactly the rational map associated to the system $|2K_S^{-1}|$, where the target space is the fiber plane $\pi^{-1}(\lambda)$. By Proposition 2.2, this means that $\Phi|_{f^{-1}(\lambda)} : f^{-1}(\lambda) \rightarrow \pi^{-1}(\lambda)$ is surjective as far as $f^{-1}(\lambda)$ is non-singular. Therefore Φ itself is also surjective to Y . Now the final claim (2 to 1 over Y) is immediate from these considerations and Proposition 2.2. \square

Thus the anticanonical map is 2 to 1 over the scroll Y . Further, by the above argument and Proposition 2.2, the branch locus of the 2 to 1 map has degree 4 on the planes $f^{-1}(\lambda)$, from which one might find similarity with those in the case of $3\mathbb{CP}^2$ [12, 16]. But we are yet far from the goal. In the next subsection we shall investigate structure of the anticanonical map more closely.

3.2. Modification of the anticanonical map. We use the notations Φ , Λ , Y , f and π given in the proof of the proposition. Further define l to be the singular locus of the scroll Y . l is a line, and is exactly the indeterminacy locus of the projection π . This line plays an important role throughout this paper. Note that for a hyperplane $H \subset \mathbb{CP}^4$, the intersection $Y \cap H$ splits to planes iff H projects to a line in \mathbb{CP}^2 (in which the conic Λ is contained), and otherwise $Y \cap H$ is a cone over Λ whose vertex is the point $l \cap H$. Further, in the former situation, $Y|_H$ is a double plane (i.e. a non-reduced plane of multiplicity 2) iff the line is tangent to Λ . Let $\nu : \tilde{Y} \rightarrow Y$ be the blowup at l , and Σ the exceptional divisor.

\tilde{Y} is biholomorphic to the total space of the \mathbb{CP}^2 -bundle $\mathbb{P}(\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}) \rightarrow \Lambda$, and Σ is identified with the subbundle $\mathbb{P}(\mathcal{O}(2)^{\oplus 2})$, so that it is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. More invariantly, we have a natural isomorphism $\Sigma \simeq l \times \Lambda$. The composition $\tilde{Y} \rightarrow Y \rightarrow \Lambda$ is a morphism which is identified with the bundle projection, for which we denote by $\tilde{\pi}$.

In order to treat the branch locus of the degree 2 morphism Φ_1 (see Proposition 3.2) properly, we consider a lifting problem of the morphism Φ_1 to \tilde{Y} . By definition of f , the composition $Z_1 \xrightarrow{\mu_1} Z \xrightarrow{f} \Lambda \subset \mathbb{CP}^2$ is the rational map associated to the 2-dimensional linear system $\mu_1^*|S^2H^0(F)|$ on Z_1 (whose members are the total transforms of members of the linear system $|S^2H^0(F)|$). This linear system on Z_1 is a subsystem of $|\mu_1^*(2F)|$. But since the pencil $|F|$ has C_1 and \overline{C}_1 as components of the base locus, all the above total transforms contain the divisor $2E_1 + 2\overline{E}_1$. Hence subtracting $E_1 + \overline{E}_1$ and recalling $\mathcal{L}_1 = \mu_1^*2F - E_1 - \overline{E}_1$, we can regard linear system $\mu_1^*|S^2H^0(F)|$ as a subsystem of $|\mathcal{L}_1|$. This 2-dimensional subsystem of $|\mathcal{L}_1|$ still has $E_1 + \overline{E}_1$ as the fixed components, so by subtracting it we obtain a 2-dimensional subsystem of $|\mathcal{L}_1 - E_1 - \overline{E}_1| = |\mu_1^*2F - 2(E_1 + \overline{E}_1)|$, which is readily seen to coincide with $|\mu_1^*2F - 2(E_1 + \overline{E}_1)|$ itself. Hence the composition $Z_1 \rightarrow Z \rightarrow \Lambda$ can be regarded as the rational map associated to $|\mu_1^*2F - 2(E_1 + \overline{E}_1)|$. However, the curves C_2 and \overline{C}_2 are contained in $\text{Bs}|F|$ (Proposition 3.1 (ii)), and the strict transforms of these curves to Z_1 are exactly the base locus of $|\mu_1^*2F - 2(E_1 + \overline{E}_1)|$. Thus the composition $Z_1 \rightarrow Z \rightarrow \Lambda$ has the strict transforms of C_2 and \overline{C}_2 as its indeterminacy locus. Therefore, the morphism $\Phi_1 : Z_1 \rightarrow Y$ cannot be lifted to $Z_1 \rightarrow \tilde{Y}$ as a morphism, because the composition with $\tilde{\pi} : \tilde{Y} \rightarrow \Lambda$ is not a morphism.

So let $\mu_2 : Z_2 \rightarrow Z_1$ be the blowup at $C_2 \cup \overline{C}_2$, and E_2 and \overline{E}_2 the exceptional divisors. Here we are regarding C_2 and \overline{C}_2 as curves in Z_1 . Let $\Phi_2 := \Phi_1 \circ \mu_2$, and $\mathcal{L}_2 := \mu_2^*\mathcal{L}_1$. (This time we do not subtract $E_2 + \overline{E}_2$ because C_2 and \overline{C}_2 are not base curves of $|\mathcal{L}_1|$.) Obviously Φ_2 is the rational map associated to $|\mathcal{L}_2|$, and it is clearly a morphism. Then we have the following:

Proposition 3.3. *The morphism $\Phi_2 : Z_2 \rightarrow Y$ can be lifted to a morphism $\tilde{\Phi}_2 : Z_2 \rightarrow \tilde{Y}$. Namely there is a morphism $\tilde{\Phi}_2 : Z_2 \rightarrow \tilde{Y}$ such that Φ_2 factors as $Z_2 \xrightarrow{\tilde{\Phi}_2} \tilde{Y} \xrightarrow{\nu} Y$.*

Proof. As in the above explanation, the composition $Z_2 \rightarrow Z_1 \rightarrow \Lambda \subset \mathbb{CP}^2$ is the rational map associated to the system $|\mu_2^*\{\mu_1^*2F - 2(E_1 + \overline{E}_1)\}|$. In the same way for the identification between a conic and a line on a plane by means of the projection from a point, once we fix any non-zero element of $H^0(\mu_1^*F - E_1 - \overline{E}_1)$, by taking a product with it, members of the system $|\mu_1^*2F - 2(E_1 + \overline{E}_1)|$ can be identified with those of the pencil $|\mu_1^*F - (E_1 + \overline{E}_1)|$, and the rational map associated to $|\mu_1^*2F - 2(E_1 + \overline{E}_1)|$ is identified with the rational map associated to $|\mu_1^*F - (E_1 + \overline{E}_1)|$. Therefore the composition $Z_2 \rightarrow Z_1 \rightarrow \Lambda \subset \mathbb{CP}^2$ can be identified with the rational map associated to the pencil $|\mu_2^*(\mu_1^*F - E_1 - \overline{E}_1)|$. This pencil has $E_2 + \overline{E}_2$ as the fixed component, and if we subtract this, the pencil becomes free, since $|F|_S = |K_S^{-1}|$ and $|K_S^{-1}|$ consists of a single member $C_1 + \overline{C}_1 + C_2 + \overline{C}_2$ (a reduced curve) by Proposition 2.1 (i). Hence the composition $Z_2 \rightarrow Z_1 \rightarrow \Lambda$ has no point of indeterminacy. We write f_2 for this morphism. Thus we are in the following left situation :

$$(3.5) \quad \begin{array}{ccc} Z_2 & & \tilde{Y} \\ f_2 \downarrow & \swarrow & \downarrow \nu \\ \Lambda & \xleftarrow{\pi} & Y \end{array} \quad \begin{array}{ccc} Z_2 & \xrightarrow{\tilde{\Phi}_2} & \tilde{Y} \\ f_2 \downarrow & \searrow \Phi_2 & \downarrow \nu \\ \Lambda & \xleftarrow{\pi} & Y \end{array}$$

where all maps except π are morphisms, and the 2 triangles are commutative. (The map $Z_2 \rightarrow Y$ is Φ_2 and the map $\tilde{Y} \rightarrow \Lambda$ is $\tilde{\pi}$.) For lifting Φ_2 to \tilde{Y} , we need to assign a point of \tilde{Y} for each point of Z_2 . For this, recall that ν is isomorphic outside $\nu^{-1}(l) = \Sigma$. For $z \in Z_2 \setminus \Phi_2^{-1}(l)$, of course, we assign the point $\nu^{-1}(\Phi_2(z))$. For $z \in \Phi_2^{-1}(l)$ define $\lambda := f_2(z)$ and $y := \Phi_2(z) \in l$. The inverse image $\nu^{-1}(y)$ is a fiber of the projection $\Sigma \rightarrow l$. In accordance with the natural isomorphism $\Sigma \simeq l \times \Lambda$, this fiber and the fiber $\tilde{\pi}^{-1}(\lambda)$ intersect at a unique point. Let $\tilde{y} \in \tilde{Y}$ be this point, and we assign \tilde{y} to z . Define $\tilde{\Phi}_2 : Z_2 \rightarrow \tilde{Y}$ to be the map thus obtained. Then $\tilde{\Phi}_2$ is clearly continuous and is a lift of Φ_2 . As $\tilde{\Phi}_2$ is holomorphic on the complement of the analytic subset $\Phi_2^{-1}(l)$, Riemann's extension theorem means that it is automatically holomorphic on the whole of Z_2 . Thus we get the situation right in (3.5) and obtained the desired lift $\tilde{\Phi}_2$. \square

Since the lift $\tilde{\Phi}_2 : Z_2 \rightarrow \tilde{Y}$ is a degree 2 morphism between non-singular spaces, we can speak about its branch divisor. Namely we first define the ramification divisor R on Z_2 as a zero divisor of a natural section (defined by the Jacobian) of the line bundle $K_{Z_2} - \tilde{\Phi}_2^* K_{\tilde{Y}}$, and then let the branch divisor \tilde{B} to be the image $\tilde{\Phi}_2(R)$, which is necessarily a divisor. We can determine the cohomology class of this divisor as follows:

Proposition 3.4. *Let \tilde{B} be the branch divisor of the lift $\tilde{\Phi}_2 : Z_2 \rightarrow \tilde{Y}$ as above. Then $\tilde{B} \in |\mathcal{O}_{\tilde{Y}}(4)|$, where $\mathcal{O}_{\tilde{Y}}(1) := \nu^* \mathcal{O}_Y(1) = \nu^* \mathcal{O}_{\mathbb{CP}^4}(1)|_Y$.*

Proof. As before let Σ be the exceptional divisor of the blowup $\tilde{Y} \rightarrow Y$, and let \mathfrak{f} be the cohomology class of the fiber class $\tilde{\pi}^* \mathcal{O}_\Lambda(1)$. The cohomology group $H^2(\tilde{Y}, \mathbb{Z})$ is a free \mathbb{Z} -module generated by Σ and \mathfrak{f} . Σ is isomorphic to $l \times \Lambda$, and the restriction $\nu|_\Sigma$ can be identified with the projection to l . Define $(0, 1)$ to be the bidegree of a fiber of this projection. Then the normal bundle is $N_{\Sigma/\tilde{Y}} \simeq \mathcal{O}(-2, 1)$, while \mathfrak{f} is restricted to the class $(1, 0)$. From these we can readily deduce that the restriction map $H^2(\tilde{Y}, \mathbb{Z}) \rightarrow H^2(\Sigma, \mathbb{Z})$ is isomorphic.

Let $H \subset \mathbb{CP}^4$ be any hyperplane containing the line l . Then since Λ is a conic, we have $\nu^{-1}(H) = \Sigma + 2\mathfrak{f}$, which means

$$(3.6) \quad \nu^* \mathcal{O}_Y(1) = \Sigma + 2\mathfrak{f} \text{ in } H^2(\tilde{Y}, \mathbb{Z}).$$

Then by using the above explicit form of the restriction map, we obtain

$$(3.7) \quad \mathcal{O}_{\tilde{Y}}(1)|_\Sigma \simeq \Sigma|_\Sigma + 2\mathfrak{f}|_\Sigma = \mathcal{O}(-2, 1) + \mathcal{O}(2, 0) = \mathcal{O}(0, 1).$$

Hence in order to prove $\tilde{B} \in |\mathcal{O}_{\tilde{Y}}(4)|$, it suffices to show $\tilde{B}|_\Sigma \in |\mathcal{O}(0, 4)|$.

In order to obtain the restriction $\tilde{B}|_\Sigma$, for each $\lambda \in \Lambda$ we write $S_\lambda := f_2^{-1}(\lambda)$, which is the strict transform of a member of the pencil $|F|$. Then it is not difficult to see that the restriction $\tilde{\Phi}_2|_{S_\lambda} : S_\lambda \rightarrow \tilde{\pi}^{-1}(\lambda) = \mathbb{CP}^2$ is naturally identified with the restriction of the original restriction $\Phi|_{S_\lambda} : S_\lambda \rightarrow \mathbb{CP}^2$. If S_λ is non-singular (which is the case for almost all λ), the last restriction $\Phi|_{S_\lambda}$ is exactly the bi-anticanonical map of S_λ . Hence by Propositions 2.2 and 2.3, the branch curve is a quartic curve which is tangent to the line l at 2 points. The last 2 points are independent of the choice of λ , since they are exactly $p_2 = \Phi(C_2)$ and $\bar{p}_2 = \Phi(\bar{C}_2)$. Therefore $\tilde{B}|_\Sigma$ contains the fibers of $\Sigma \rightarrow l$ (the restriction of $\tilde{Y} \rightarrow Y$ to Σ) over the points p_2 and \bar{p}_2 by multiplicity 2 respectively. Moreover since the intersection of the line l with the branch quartic curve of $S_\lambda \rightarrow \mathbb{CP}^2$ consists of the 2 points p_2 and \bar{p}_2 , if $\tilde{B}|_\Sigma$ contains an irreducible curve of bidegree (k, l) with $k > 0$, then we have $(k, l) = (1, 0)$.

But this cannot occur since the original Φ does not contain l as a branch locus. Thus we have obtained $\tilde{B}|_\Sigma \in |\mathcal{O}(0, 4)|$. \square

Define a divisor B on Y by $B := \nu(\tilde{B})$. By Proposition 3.4, $B \in |\mathcal{O}_Y(4)|$, and since $B = \nu(\tilde{\Phi}_2(R)) = \Phi_2(R)$, the morphism $\Phi_2 : Z_2 \rightarrow Y$ is a generically 2 to 1 covering with branch B . (The former implies that B is a cut of Y by a quartic hypersurface. We will explicitly obtain a defining equation of this hypersurface in the next section.) However Φ_2 is *not* a finite map but contracts the divisors E_1, \overline{E}_1, E_2 and \overline{E}_2 as we see next until completing as Proposition 3.5. For this, we first notice that the exceptional divisors E_2 and \overline{E}_2 of $\mu_2 : Z_2 \rightarrow Z_1$ are isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the normal bundles satisfy

$$(3.8) \quad N_{E_2/Z_2} \simeq \mathcal{O}(-1, -1), \quad N_{\overline{E}_2/Z_2} \simeq \mathcal{O}(-1, -1).$$

(See Figure 3 (c).) Therefore E_2 and \overline{E}_2 can also be blowdown along the projection different from the original $E_2 \rightarrow C_2$ and $\overline{E}_2 \rightarrow \overline{C}_2$. Let $\mu_3 : Z_2 \rightarrow Z_3$ be this blowdown. (See (c) \rightarrow (d) in Figure 3.) Z_3 is still non-singular. The birational transformation from Z_1 to Z_3 is exactly Atiyah's flop at C_2 and \overline{C}_2 . The divisors E_1 and \overline{E}_1 in Z_1 are also isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, and they are respectively blown up at 2 points through μ_2 . We use the same letters E_1 and \overline{E}_1 to mean these divisors in Z_2 . These divisors are not affected by the blowdown μ_3 , and we still denote by E_1 and \overline{E}_1 for their images in Z_3 , as displayed in Figure 3 (d).

Next we show that these 2 divisors E_1 and \overline{E}_1 in Z_3 can be contracted to non-singular rational curves simultaneously. For this we first consider the divisor E_1 in Z_1 , so that $E_1 \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, and take the cohomology class of a fiber of the projection to \mathbb{CP}^1 which is different from the projection to C_1 . Next pullback the class by the blowup μ_2 and push it to $E_1 \subset Z_3$ by μ_3 . (In Figure 3 these cohomology classes are represented by non-dotted lines on E_1 .) Thus we obtain a cohomology class on $E_1 \subset Z_3$ whose self-intersection number is zero. The linear system on this E_1 having this cohomology class is clearly a free pencil, and induces a morphism to \mathbb{CP}^1 . Let $g : E_1 \rightarrow \mathbb{CP}^1$ be this morphism. General fibers of g are non-singular rational curves, and there exist precisely 2 singular fibers, both of which consist of 2 non-singular rational curves intersecting at a point. The same is true for \overline{E}_1 , and let $\overline{g} : \overline{E}_1 \rightarrow \mathbb{CP}^1$ the morphism corresponding to g . Now g and \overline{g} naturally fit on the intersection $E_1 \cap \overline{E}_1$ and form a morphism $g \cup \overline{g} : E_1 \cup \overline{E}_1 \rightarrow \mathbb{CP}^1 \cup \mathbb{CP}^1$. Here note that these two \mathbb{CP}^1 -s are identified at 2 points, and $g \cup \overline{g}$ has reducible fibers exactly over these 2 points, both of which consist of *three* rational curves. In Figure 3 (d), these 2 reducible fibers are written by 3 bold lines respectively.

We are going to show that the reducible connected divisor $E_1 \cup \overline{E}_1$ on Z_3 can be contracted along $g \cup \overline{g}$. For this we need to examine the normal bundle, $[E_1 + \overline{E}_1]|_{E_1 \cup \overline{E}_1}$. The restriction of the normal bundle $N_{E_1/Z_3} = [E_1]|_{E_1}$ is degree (-2) on irreducible fibers of g , and degree (-1) on the 2 irreducible components of the (two) singular fibers. (See Figure 3 (d).) From this we deduce that the restriction of the line bundle $[E_1 + \overline{E}_1]|_{E_1 \cup \overline{E}_1}$ is (-2) on irreducible fibers of $g \cup \overline{g}$, and (-1) on the end components of the reducible fibers, while $(-1) + (-1) = -2$ on the middle component of the reducible fibers. Now by the relative version of Nakai-Moishezon criterion for ampleness, these numerical data imply that the dual line bundle $[E_1 + \overline{E}_1]^*|_{E_1 \cup \overline{E}_1}$ is $(g \cup \overline{g})$ -ample. Moreover again from the numerical data, for a direct image of the dual bundle, we have

$$R^1(g \cup \overline{g})_*([E_1 + \overline{E}_1]^*|_{E_1 \cup \overline{E}_1})^{\otimes m} = 0 \text{ for any } m > 0.$$

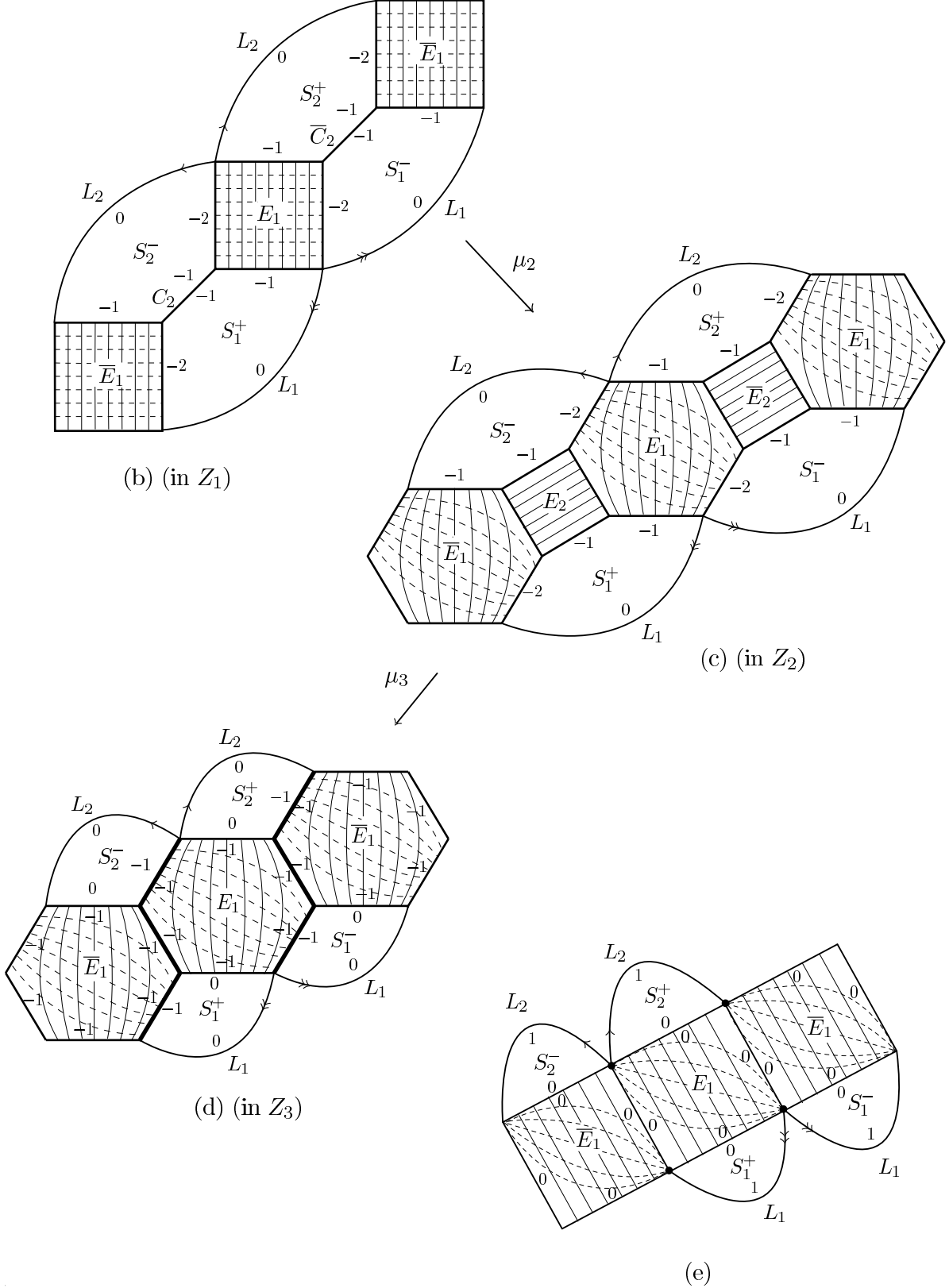


FIGURE 3. The transformations from Z_1 to Z_3 . The picture (b) is identical to (b) in Figure 1; the present (b) is obtained from the original (b) by just cutting out (just for presentation) along the 2 twistor lines L_1 and L_2 . So in each of (b), (c) and (d), the two L_1 -s are identified in the direction indicated by the arrows, and the same for L_2 -s. (e) is obtained from (d) by contracting four $(-1, -1)$ -curves, and will be used later.

Therefore by a theorem of Fujiki [4, Theorem 2], the divisor $E_1 \cup \overline{E}_1$ can be contracted to $\mathbb{CP}^1 \cup \mathbb{CP}^1$ along the morphism $g \cup \overline{g}$. Let $\mu_4 : Z_3 \rightarrow Z_4$ be the birational morphism obtained this way, and put $l_4 := \mu_4(E_1)$ and $\overline{l}_4 := \mu_4(\overline{E}_1)$, so that $l_4 \cup \overline{l}_4$ can be naturally identified with the target space of $g \cup \overline{g}$. As the degree of the restriction of N_{E_1/Z_3} to irreducible fibers of g is (-2) as above, we have $\text{Sing } Z_4 = l_4 \cup \overline{l}_4$, and possibly outside the 2 points $l_4 \cap \overline{l}_4$, Z_4 has ordinary double points along $l_4 \cup \overline{l}_4$. (In Section 6 we will obtain an explicit defining equation of Z_4 around the 2 points.) This way by contracting $E_1 \cup \overline{E}_1$ in Z_3 we have obtained a singular variety Z_4 . Then the morphism Φ_2 descends to Z_4 :

Proposition 3.5. *Let $\Phi_2 : Z_2 \rightarrow Y$ be the generically 2 to 1 covering as before. Then Φ_2 descends to a morphism $Z_4 \rightarrow Y$. Namely there is a morphism $\Phi_4 : Z_4 \rightarrow Y$ such that $\Phi_4 \circ \mu_4 \circ \mu_3 = \Phi_2$.*

Proof. We first show that Φ_2 descends to a morphism $\Phi_3 : Z_3 \rightarrow Y$. Recall that Φ_2 is induced by the system $|\mathcal{L}_2|$, where $\mathcal{L}_2 = \mu_2^* \mathcal{L}_1$ and $\mathcal{L}_1 = \mu_1^* 2F - E_1 - \overline{E}_1$. In accordance with those on Σ_1 in the proof of Proposition 3.4, let $(0, 1)$ be the fiber class of the projection $E_1 \rightarrow C_1$. Then we obtain $\mathcal{L}_1|_{E_1} \simeq 2\mu_1^*(F|_{C_1}) - N_{E_1/Z_1} \simeq 2(\mu_1^* K_S^{-1}|_{C_1}) - \mathcal{O}(-1, -2) \simeq 2\mu_1^* \mathcal{O}_{C_1}(-1) + \mathcal{O}(1, 2) \simeq \mathcal{O}(1, 0)$. (See Figure 3 (b) for $N_{E_1/Z_1} \simeq \mathcal{O}(-1, -2)$.) Further, as the curve $C_2 \subset Z_1$ intersects E_1 transversally at exactly 1 point and the same for \overline{E}_1 (again see Figure 3 (b)), we have $\mathcal{L}_1|_{C_2} \simeq (\mu_1^* 2F)|_{C_2} - \mathcal{O}_{C_2}(2) \simeq 2F|_{C_2} - \mathcal{O}_{C_2}(2) \simeq 2K_S^{-1}|_{C_2} - \mathcal{O}_{C_2}(2) \simeq \mathcal{O}_{C_2}$. Therefore pulling back to Z_2 we obtain $\mathcal{L}_2|_{E_1} \simeq \mu_2^* \mathcal{O}(1, 0)$, $\mathcal{L}_2|_{E_2} \simeq \mathcal{O}_{E_2}$, and analogous result for the restrictions to \overline{E}_1 and \overline{E}_2 . These imply that the direct image sheaf $(\mu_3)_* \mathcal{L}_2 =: \mathcal{L}_3$ is still invertible and $\mathcal{L}_3|_{E_1} \simeq \mu_2^* \mathcal{O}_{E_1}(1, 0)$. If we use the projection $g : E_1 \rightarrow l_4$, the last isomorphism can be rewritten as

$$(3.9) \quad \mathcal{L}_3|_{E_1} \simeq g^* \mathcal{O}_{l_4}(1).$$

Then since $\mathcal{L}_2 \simeq \mu_3^* \mathcal{L}_3$, the morphism associated to $|\mathcal{L}_2|$ factors through the morphism associated to $|\mathcal{L}_3|$. Letting Φ_3 be the last morphism, this means $\Phi_2 = \Phi_3 \circ \mu_3$ as claimed.

In a similar way we next show that Φ_3 descends to a morphism $\Phi_4 : Z_4 \rightarrow Y$. From (3.9) the direct image $(\mu_4)_* \mathcal{L}_3 =: \mathcal{L}_4$ is still an invertible sheaf (on Z_4), whose restriction to l_4 is of degree 1. Then by the natural isomorphisms $H^0(Z_2, \mathcal{L}_2) \simeq H^0(Z_3, \mathcal{L}_3) \simeq H^0(Z_4, \mathcal{L}_4)$ the map associated to $|\mathcal{L}_3|$ factors through the map induced by $|\mathcal{L}_4|$. Therefore if we define Φ_4 to be the map associated to $|\mathcal{L}_4|$, we have $\Phi_3 = \Phi_4 \circ \mu_4$, as claimed. Hence we have obtained $\Phi_2 = \Phi_3 \circ \mu_3 = \Phi_4 \circ \mu_4 \circ \mu_3$. \square

Thus we arrived at the following situation:

$$(3.10) \quad \begin{array}{ccc} Z_2 & \xrightarrow{\mu_4 \circ \mu_3} & Z_4 \\ \mu_2 \downarrow & \searrow \Phi_2 & \downarrow \Phi_4 \\ Z_1 & \xrightarrow{\Phi_1} & Y \end{array}$$

where all maps are morphisms and the 2 triangles are commutative. Since $\mu_4 \circ \mu_3$ is birational, Φ_4 is still a degree 2 morphism branching at the divisor B . But in contrast with Φ_2 , it does not contract divisors anymore:

Proposition 3.6. *The morphism Φ_4 does not contract any divisor to a point or a curve.*

Proof. It is enough to show that the morphism $\Phi_1 : Z_1 \rightarrow Y$ does not contract any irreducible divisor other than E_1 and \overline{E}_1 . Let D be such a divisor. If D is real, then

$D \in |\mu_1^*(kF) - lE_1 - l\overline{E}_1|$ for some $k \geq 1$ and $l \geq 0$, and $(\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot D = 0$ by the contractedness property. For computing this intersection number, we notice, as $\mu_1^*F|_{E_1} \simeq \mu_1^*(K_S^{-1}|_{C_1}) \simeq \mathcal{O}_{E_1}(0, -1)$, that we have $(\mu_1^*F)^2 \cdot E_1 = (\mu_1^*F|_{E_1})^2 = 0$. Hence we have

$$\begin{aligned} (\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot E_1 &= 4\mu_1^*F^2 \cdot E_1 - 4\mu_1^*F \cdot (E_1 + \overline{E}_1) \cdot E_1 + (E_1 + \overline{E}_1)^2 \cdot E_1 \\ &= 4 \cdot 0 - 4\mu_1^*F \cdot E_1^2 + E_1^3 \\ &= -4\mathcal{O}_{E_1}(0, -1) \cdot \mathcal{O}_{E_1}(-1, -2) + \mathcal{O}_{E_1}(-1, -2)^2 \\ &= -4 + 4 = 0, \end{aligned}$$

and the same for \overline{E}_1 . From these, recalling $F^3 = 0$ (as we are over $4\mathbb{CP}^2$), we obtain

$$\begin{aligned} (\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot D &= (\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot (\mu_1^*(kF) - lE_1 - l\overline{E}_1) \\ &= (\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot \mu_1^*(kF) \\ &= E_1^2 \cdot \mu_1^*(kF) + \overline{E}_1^2 \cdot \mu_1^*(kF) - 4k\mu_1^*F^2 \cdot (E_1 + \overline{E}_1) \\ &= 2N_{E_1/Z_1} \cdot \mu_1^*(kF)|_{E_1} \\ &= 2k\mathcal{O}_{E_1}(-1, -2) \cdot \mathcal{O}_{E_1}(0, -1) = 2k. \end{aligned}$$

Therefore we have $(\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot D > 0$. Hence D is not contracted to a curve or a point by Φ_1 . When D is not real, by applying the above computations for $D + \overline{D}$ instead of D , we obtain $(\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot (D + \overline{D}) > 0$. Hence, since $(\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot D = (\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot \overline{D}$, we again conclude $(\mu_1^*2F - E_1 - \overline{E}_1)^2 \cdot D > 0$. Hence in the non-real case too, D cannot be contracted to a point or a curve by Φ_1 , as claimed. \square

As a consequence, we obtain the following

Proposition 3.7. *The branch divisor B has only isolated singularities.*

Proof. Let $Z_4 \xrightarrow{\mu_5} Z_5 \xrightarrow{\Phi_5} Y$ be the Stein factorization of the morphism Φ_4 . μ_5 is necessarily birational. Then Φ_5 is just a double covering with branch B . Hence if B has singularities along a curve, so is Z_5 . Since B does not contain l and $\text{Sing } Z_4 = l_4 \cup \overline{l}_4$, this means that the birational morphism μ_5 resolves the singularities along the curve. Hence μ_5 contracts a divisor. This contradicts Proposition 3.6. \square

We note that the proof of Proposition 3.6 means that the original anticanonical map $\Phi : Z \rightarrow Y$ does not contract any divisor. We also note that the morphism μ_5 in the proof of Proposition 3.7 contracts (at worst) finitely many curves, and all these curves are over singular points of B . We will investigate these singularities in detail in Section 4.4.

4. DEFINING EQUATION OF THE BRANCH QUARTIC HYPERSURFACE

In the last section we analyzed the anticanonical system on the twistor space in detail and obtained the space Z_4 and a degree 2 morphism $\Phi_4 : Z_4 \rightarrow Y$ which is explicitly birational to the original anticanonical map, and which does not contract any divisor. We further showed that the branch divisor B of Φ_4 is a cut of Y by a quartic hypersurface in \mathbb{CP}^4 . In this section we shall determine defining equation of this quartic hypersurface. We also determine the number of singularities of the branch divisor.

4.1. Finding double curves on B . Our way for obtaining the equation includes finding hyperplanes $H \subset \mathbb{CP}^4$ such that, regarding the intersection $H \cap Y$ (which is either a plane or a cone as in the beginning of Section 3.2) as a reduced divisor on Y , the restriction $B|_{H \cap Y}$ is a double curve (i.e. a non-reduced curve of multiplicity 2). So it is similar to the method of Poon [16] (for the case of $3\mathbb{CP}^2$), but the origin of some of the double curves is different from the case of $3\mathbb{CP}^2$.

We keep the notations from the last section. We are going to show the existence of *five* double curves, two of which are easy to find as we see now. First we recall there are diagrams in (3.4); Λ is a conic in \mathbb{CP}^2 and for each $\lambda \in \Lambda$, $S_\lambda := f^{-1}(\lambda)$ is a member of the pencil $|F|$, i.e. Λ is a parameter space of $|F|$. For any $\lambda \in \Lambda$, $\pi^{-1}(\lambda)$ is a plane containing the line l . Let $S_1 = S_1^+ + S_1^-$ and $S_2 = S_2^+ + S_2^-$ be the reducible members as in (3.1), and let 0 and ∞ be the points of Λ such that $f^{-1}(0) = S_1$ and $f^{-1}(\infty) = S_2$ hold. Let H_1 and H_2 be the hyperplanes in \mathbb{CP}^4 which are the inverse images of the tangent line of Λ at the points 0 and ∞ respectively under the projection π . Then the restrictions $H_1|_Y$ and $H_2|_Y$ are double planes, and we have $\Phi^{-1}(H_1) = 2S_1$ and $\Phi^{-1}(H_2) = 2S_2$. Letting $L_1 = S_1^+ \cap S_1^-$ and $L_2 = S_2^+ \cap S_2^-$ be the twistor lines as before, we define

$$(4.1) \quad \mathcal{C}_1 := \Phi(L_1) \text{ and } \mathcal{C}_2 := \Phi(L_2).$$

Then since Φ is a real map and 0 and ∞ are real points, and since Φ does not contract any divisor by Proposition 3.6, S_1^+ and S_1^- are mapped birationally to the plane $\pi^{-1}(0)$ and S_2^+ and S_2^- are mapped birationally to the plane $\pi^{-1}(\infty)$. Hence \mathcal{C}_1 and \mathcal{C}_2 are contained in the branch divisor B in such a way that the restrictions $B|_{H_1 \cap Y}$ and $B|_{H_2 \cap Y}$ respectively contain \mathcal{C}_1 and \mathcal{C}_2 by multiplicity 2. But as we know that B is a cut of Y by a hyperquartic surface by Proposition 3.4, \mathcal{C}_1 and \mathcal{C}_2 must be conics. So we call these two *double conics*.

These double conics are analogous to the 4 conics contained in the coordinates tetrahedron in \mathbb{CP}^3 used in [16] in the case of $3\mathbb{CP}^2$, but in the present case there are only 2 since there are only 2 reducible members of $|F|$. Next we find other 3 double curves. For this recall that our twistor space Z contains the surface S constructed in Section 2 as a real member of $|F|$. Take up the blowup $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ which was concretely given when constructing S , and let e_i and \bar{e}_i , $1 \leq i \leq 4$, be the exceptional curves of the blowup. Here we take indices such that $e_i \cdot C_1 = 1$ for $i = 1, 2, 3$ and $e_4 \cdot C_2 = 1$. Next let $\{\alpha_i \mid 1 \leq i \leq 4\}$ be an orthonormal basis of $H^2(4\mathbb{CP}^2, \mathbb{Z})$ determined by $\{e_i\}$; namely letting $t : Z \rightarrow 4\mathbb{CP}^2$ be the twistor fibration, $t^*\alpha_i|_S = e_i - \bar{e}_i$ in $H^2(S, \mathbb{Z})$. Then we have the following proposition which is technically significant for our purpose:

Proposition 4.1. *For each i with $1 \leq i \leq 3$ (not 4), the linear systems $|F + t^*\alpha_i|$ and $|F - t^*\alpha_i|$ consist of a single element. Moreover, all these 6 divisors are irreducible.*

Proof. In this proof for simplicity we write α_i for $t^*\alpha_i$. It is enough to prove the claim for the system $|F + \alpha_i|$. Fix any i with $1 \leq i \leq 3$. We have $(F + \alpha_i)|_S = K_S^{-1} + (e_i - \bar{e}_i) = \epsilon^*\mathcal{O}(2, 2) - \sum_{j=1}^4 (e_j + \bar{e}_j) + (e_i - \bar{e}_i)$, which can be rewritten as

$$\left(\epsilon^*\mathcal{O}(1, 0) - \sum_{1 \leq j \leq 3} \bar{e}_j \right) + \left(\epsilon^*\mathcal{O}(1, 2) - \bar{e}_i - e_4 - \bar{e}_4 - \sum_{1 \leq j \leq 3, j \neq i} e_j \right).$$

The intersection number between $\bar{C}_1 \sim \epsilon^*\mathcal{O}(1, 0) - \sum_{j=1}^3 \bar{e}_j$ and the above $(F + \alpha_i)|_S$ is easily computed to be (-2) . Hence \bar{C}_1 is a fixed component of $|(F + \alpha_i)|_S|$. Further counting dimension, the remaining system $|\epsilon^*\mathcal{O}(1, 2) - \bar{e}_i - e_4 - \bar{e}_4 - \sum_{j=1, j \neq i}^3 e_j|$ consists of a single member, which is the strict transform of a $(1, 2)$ -curve passing through the 5 points

$\bar{q}_i, q_4, \bar{q}_4$ and q_j with $1 \leq j \leq 3$ and $j \neq i$, where $q_i = \epsilon(e_i)$ and $\bar{q}_i = \epsilon(\bar{e}_i)$. Thus we have $h^0((F + \alpha_i)|_S) = 1$. On the other hand from Riemann-Roch formula and Hitchin's vanishing theorem [6] we deduce $H^1(Z, \alpha_i) = 0$. Then by the standard exact sequence $0 \rightarrow \alpha_i \rightarrow F + \alpha_i \rightarrow (F + \alpha_i)|_S \rightarrow 0$, we obtain the exact sequence $0 \rightarrow H^0(Z, \alpha_i) \rightarrow H^0(Z, F + \alpha_i) \rightarrow H^0(S, (F + \alpha_i)|_S) \rightarrow 0$. As $H^0(Z, \alpha_i) = 0$, we obtain $H^0(Z, F + \alpha_i) \simeq H^0(S, (F + \alpha_i)|_S)$. Hence we get $H^0(Z, F + \alpha_i) \simeq \mathbb{C}$.

For the irreducibility, all the divisors S_1^+, S_1^-, S_2^+ and S_2^- (the irreducible components of reducible members of $|F|$) are degree 1 on Z . Moreover, it is not difficult to show that the Chern classes of these divisors are given by *the half of* the following classes:

$$(4.2) \quad F - \sum_{1 \leq j \leq 4} \alpha_j, \quad F + \sum_{1 \leq j \leq 4} \alpha_j, \quad F + \alpha_4 - \sum_{1 \leq j \leq 3} \alpha_j, \quad F - \alpha_4 + \sum_{1 \leq j \leq 3} \alpha_j.$$

Then it is a easy to check that a sum of any two of these 4 classes (allowing to choose the same one) are not equal to $2(F + \alpha_i)$, for any $1 \leq i \leq 3$. This implies the desired irreducibility. (On the other hand, by (4.2), the systems $|F \pm \alpha_4|$ are also non-empty, but both of them consist of a single *reducible* member.) \square

In the following for $1 \leq i \leq 3$ we denote by X_i for the unique member of $|F + t^* \alpha_i|$. Then $\bar{X}_i \in |F - t^* \alpha_i|$, and $X_i + \bar{X}_i \in |2F|$. Thus we obtained 3 reducible real members of the anticanonical system on Z . We remark that from the proof of Proposition 4.1 these 3 members originally come from the choice of 3 points on C_1 in the construction of S at the beginning of Section 2. By using these we obtain a special basis of $H^0(2F) \simeq \mathbb{C}^5$ as follows:

Proposition 4.2. *For any i with $1 \leq i \leq 3$, let $\xi_i \in H^0(Z, 2F)$ be an element such that $(\xi_i) = X_i + \bar{X}_i$. Then $S^2 H^0(Z, F) (\simeq \mathbb{C}^3)$ and any two among $\{\xi_i | 1 \leq i \leq 3\}$ generate $H^0(2F) (\simeq \mathbb{C}^5)$.*

Proof. Let $S \in |F|$ be any real irreducible member and take $s_0 \in H^0(F)$ with $(s_0) = S$. Let $s_1 \in H^0(F)$ be any element satisfying $s_1 \notin \mathbb{C}s_0$. Then $\{s_0, s_1\}$ is a basis of $H^0(F)$ and $\{s_0^2, s_0 s_1, s_1^2\}$ is a basis of $S^2 H^0(F)$. We consider the exact sequence

$$(4.3) \quad 0 \longrightarrow H^0(F) \xrightarrow{\otimes s_0} H^0(2F) \longrightarrow H^0(2K_S^{-1}) \longrightarrow 0$$

appeared in the proof of Proposition 3.2. For proving the claim of the proposition, it suffices to show that for any subset $\{i, j\} \subset \{1, 2, 3\}$, the images of s_1^2, ξ_i, ξ_j by the restriction map to S generate $H^0(2K_S^{-1})$. For this, the divisor $(s_1^2|_S)$ is exactly $2C$, where C is the unique anticanonical curve (i.e. the cycle of 4 rational curves). On the other hand we have $(\xi_i|_S) = X_i|_S + \bar{X}_i|_S$, and from the proof of Proposition 4.1 we know the curves $X_i|_S$ and $\bar{X}_i|_S$ in concrete forms, and it is not difficult to verify that the 3 bi-anticanonical curves $2C, X_i|_S + \bar{X}_i|_S, X_j|_S + \bar{X}_j|_S$ are linearly independent. Hence the 3 images generate $H^0(2K_S^{-1})$. \square

In the sequel for obtaining nice coordinates, we choose a slightly different basis $\{u_1, u_2\}$ of $H^0(Z, F)$ as follows. Namely respecting the reducible members, we choose those satisfying $(u_1) = S_1$ and $(u_2) = S_2$. By Proposition 4.2 the collection $\{u_1 u_2, u_1^2, u_2^2, \xi_1, \xi_2\}$ is a basis of $H^0(Z, 2F)$. The target space of the anticanonical map $\Phi : Z \rightarrow \mathbb{CP}^4$ is nothing but the dual projective space $\mathbb{P}H^0(Z, 2F)^* (\simeq \mathbb{CP}^4)$, and if we put

$$(4.4) \quad z_0 := u_1 u_2, \quad z_1 := u_1^2, \quad z_2 := u_2^2, \quad z_3 := \xi_1, \quad z_4 := \xi_2,$$

then $(z_0, z_1, z_2, z_3, z_4)$ can be used as homogeneous coordinates on it. (Here we remark that there is no special reason to choose ξ_1 and ξ_2 . Any choice of two among $\{\xi_1, \xi_2, \xi_3\}$ leads to the same description below.) As $\mathbb{CP}^4 = \mathbb{P}H^0(2F)^*$ as above, \mathbb{CP}^4 is equipped with a real structure and by (4.4) it is just the complex conjugation with respect to the above coordinates. In these coordinates the scroll $Y = \Phi(Z)$ is explicitly defined by the equation

$$(4.5) \quad z_0^2 = z_1 z_2,$$

and the ridge l of Y is given by $z_0 = z_1 = z_2 = 0$. For each $0 \leq i \leq 4$ we define a hyperplane by $H_i := \{z_i = 0\}$. Obviously $l \subset H_i$ for $i = 0, 1, 2$, and $l \not\subset H_i$ for $i = 3, 4$. In particular $H_1|_Y$ and $H_2|_Y$ are double planes, $H_0|_Y$ is the sum of these 2 planes, and $H_3|_Y$ and $H_4|_Y$ are cones whose vertices are the points $H_3 \cap l$ and $H_4 \cap l$ respectively.

Let $z_5 \in H^0(Z, 2F)$ be an element such that $(z_5) = X_3 + \overline{X}_3$. Then by Proposition 4.2 we can write

$$(4.6) \quad z_5 = a_0 z_0 + a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4$$

for some $a_i \in \mathbb{R}$. Let $H_5 := \{z_5 = 0\}$. Since $z_5 \notin S^2 H^0(Z, F)$ clearly, $H_5|_Y$ is also a cone.

Now the following proposition provides the promised 3 double curves on B :

Proposition 4.3. *For $i = 3, 4, 5$, the intersection of the branch divisor B with the cone $H_i \cap Y$ is a double curve of B .*

Proof. Let i be any one of 3, 4, 5. By definition of the hyperplane H_i , we have $\Phi^{-1}(H_i) = X_{i-2} + \overline{X}_{i-2}$. Since Φ does not contract any divisor by Proposition 3.6, this implies $\Phi(X_{i-2}) = \Phi(\overline{X}_{i-2}) = H_i \cap Y$. As Φ is degree 2, this means that the restrictions $\Phi|_{X_{i-2}}$ and $\Phi|_{\overline{X}_{i-2}}$ are birational over the cone $H_i|_Y$. (In particular, the non-real degree 2 divisors X_{i-2} and \overline{X}_{i-2} are birational to a cone.) Therefore the curve $X_{i-2} \cap \overline{X}_{i-2}$ is the ramification divisor of the restriction $\Phi|_{\Phi^{-1}(H_i)} : \Phi^{-1}(H_i) \rightarrow H_i \cap Y$, and hence

$$\mathcal{C}_3 := \Phi(X_1 \cap \overline{X}_1) \quad \mathcal{C}_4 := \Phi(X_2 \cap \overline{X}_2) \quad \text{and} \quad \mathcal{C}_5 := \Phi(X_3 \cap \overline{X}_3)$$

are branch divisors when restricted to $\Phi^{-1}(H_i)$. This implies the claim of the proposition. \square

As in the proof, we use the letters $\mathcal{C}_3, \mathcal{C}_4$ and \mathcal{C}_5 to mean the 3 double curves in the proposition. Then because we know that B is a cut of Y by a quartic hypersurface, we have $\mathcal{C}_i \in |\mathcal{O}_{Y \cap H_i}(2)|$, where $\mathcal{O}_{Y \cap H_i}(2) := \mathcal{O}_{\mathbb{CP}^4}(2)|_{Y \cap H_i}$. Namely, $\mathcal{C}_3, \mathcal{C}_4$ and \mathcal{C}_5 are intersection of the cone $Y \cap H_i$ with a quadratic in $H_i = \mathbb{CP}^3$. From this it follows that these 3 curves are of degree 4 in \mathbb{CP}^4 . So in the following we call these double curves *double quartic curves*. From our choice of the coordinates, any double curves can be written as , as sets,

$$(4.7) \quad \mathcal{C}_i = B \cap H_i, \quad 1 \leq i \leq 5.$$

Since B and H_i are real, all \mathcal{C}_i -s are real curves.

4.2. Quadratic hypersurfaces containing the double curves. In this section we show that there exists a quadratic hypersurface in \mathbb{CP}^4 which contains the double conics $\mathcal{C}_1, \mathcal{C}_2$ and the double quartic curves $\mathcal{C}_3, \mathcal{C}_4$ and \mathcal{C}_5 , and also show that such hyperquadric is unique up to the defining equation of the scroll Y .

First we make it clear how the 5 double curves of B intersect each other. For this for each $0 \leq i \leq 4$ we denote by $e_i \in \mathbb{CP}^4$ for the point whose coordinates are zero except z_i -component, and define some lines as follows: for each pair (i, j) with $i = 1, 2$

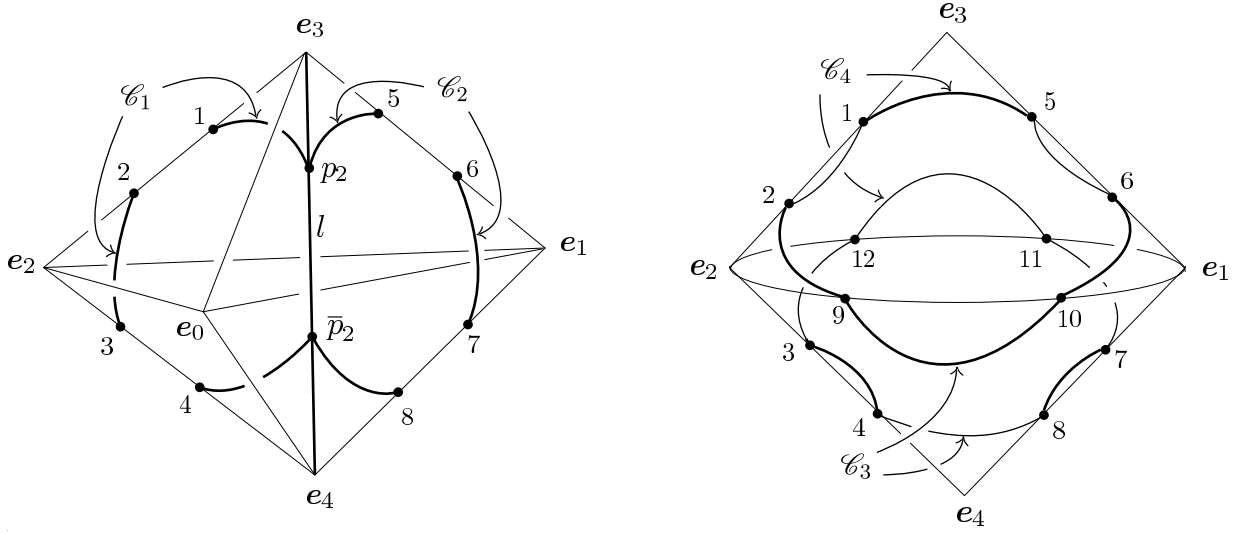


FIGURE 4. The double curves of the branch divisor B , except \mathcal{C}_5 . Both pictures lie on the same \mathbb{CP}^4 , and the common numbers represent the same point, all of which are ordinary double points of B . The left picture indicates all coordinate hyperplanes, 2-planes and lines, as well as double conics. In the right picture the upper and lower halves are the cones $Y \cap H_4$ and $Y \cap H_3$ respectively, on which the double quartic curves \mathcal{C}_4 and \mathcal{C}_3 lie.

and $j = 3, 4, 5$, define $l_{ij} := H_i \cap Y \cap H_j$. Since $H_i \cap Y$ is a (double) plane for $i = 1, 2$, this is a line. Thus we get 6 lines. If $j \neq 5$, these are coordinate lines and in Figure 4, $l_{14} = \overline{e_2 e_3}$, $l_{13} = \overline{e_2 e_4}$, $l_{24} = \overline{e_1 e_3}$, $l_{23} = \overline{e_1 e_4}$. (We do not write pictures of l_{15} and l_{25} because these are not coordinate lines. But this is just a matter of a choice of coordinates and these two play the same role as other 4 lines.) Also, for the ridge l we have $l = \overline{e_3 e_4}$ (the bold line on the left picture in Figure 4).

Then since the intersection of the branch divisor B with the plane $H_i \cap Y = \{z_0 = z_i = 0\}$ ($i = 1, 2$) is the double conics \mathcal{C}_i and since the line l_{ij} ($3 \leq j \leq 5$) is contained in this plane, the intersection $B \cap l_{ij}$ consists of (not 4 but) 2 points, and $B \cap l_{ij} = \mathcal{C}_i \cap l_{ij}$. Moreover, as $\mathcal{C}_i = \Phi(L_i)$, these 2 points cannot be identical. Thus for each of the 6 lines l_{ij} , $B \cap l_{ij}$ consists of 2 points. (In Figure 4 these points are represented by numbered points $1, 2, \dots, 7, 8$.) On the other hand, we have $B \cap H_j = \mathcal{C}_j$. Hence as $l_{ij} \subset H_j$, we obtain $B \cap l_{ij} \subset \mathcal{C}_j \cap l_{ij}$ for $i = 1, 2$ and $j = 3, 4, 5$. But since \mathcal{C}_j is an intersection of $Y \cap H_j$ with a quadric surface in H_j , $\mathcal{C}_i \cap l_{ij}$ consists at most 2 points. Hence we have the coincidence $B \cap l_{ij} = \mathcal{C}_j \cap l_{ij}$ for these i and j . Therefore we have $B \cap l_{ij} = \mathcal{C}_i \cap \mathcal{C}_j$ for $i = 1, 2$ and $j = 3, 4, 5$. By a similar reason, the intersection $B \cap l$ also consists of 2 points, which are exactly $\mathcal{C}_1 \cap \mathcal{C}_2$. In Figure 4 these points are displayed as p_2 and \bar{p}_2 . On the other hand, for each pair (j, k) with $3 \leq j < k \leq 5$ we define a plane P_{jk} by $P_{jk} = H_j \cap H_k$. Then since $B \cap H_j$ is contained in a quadric surface, and $Y \cap P_{jk}$ is a conic, $B \cap P_{jk}$ ($3 \leq j < k \leq 5$) consists of 4 points, and it coincides with $\mathcal{C}_j \cap \mathcal{C}_k$. In Figure 4, for the case $(j, k) = (3, 4)$, these are represented by numbered points 9, 10, 11, 12. (For avoiding confusion we do not write a picture for $\mathcal{C}_3 \cap \mathcal{C}_5$ and $\mathcal{C}_4 \cap \mathcal{C}_5$. The way how these curves intersect is completely analogous to that of \mathcal{C}_3 and \mathcal{C}_4 .) We list all these intersections:

- 2 points $\mathcal{C}_1 \cap \mathcal{C}_2$, which are exactly p_2 and \bar{p}_2 ,

- 12 points $\mathcal{C}_i \cap \mathcal{C}_j$ with $i = 1, 2$ and $j = 3, 4, 5$,
- 12 points $\mathcal{C}_3 \cap \mathcal{C}_4$, $\mathcal{C}_3 \cap \mathcal{C}_5$ and $\mathcal{C}_4 \cap \mathcal{C}_5$.

Collecting these, we obtain 26 points in total. Since all the double curves are the image of curves in Z by a map which is degree 1 on these curves, these 26 points form 13 conjugate pairs. Among these 26 points, the 2 points $\mathcal{C}_1 \cap \mathcal{C}_2$ are on the singular locus l of Y , and other 24 points are ordinary double points of B . (In some sense these 24 points are analogous to the 12 ordinary double points of the branch quartic surface appeared in [16] and [12].) In Section 4.4, we will show that the 2 points p_2 and \bar{p}_2 are A_3 -singular points of B .

With these situation in hand, we next show the existence of a hyperquadric which contains all the double conics:

Proposition 4.4. *There exists a real quadratic hypersurface in \mathbb{CP}^4 which contains all the 5 double curves \mathcal{C}_i , $1 \leq i \leq 5$, and which is different from the scroll Y . Moreover, such a hyperquadric is unique in the following sense: if Q and Q' are defining quadratic polynomials of two such hyperquadrics, then there exists $(c, c') \in \mathbb{R}^2$ with $(c, c') \neq (0, 0)$ such that $cQ - c'Q' \in (z_0^2 - z_1z_2)$. (Note that since the scroll Y contains all the double curves, presence of this ambiguity is obvious from the beginning.)*

Proof. As we have already seen, for $i = 3, 4$ the intersection $H_i \cap Y$ is a quadratic cone in $H_i = \mathbb{CP}^3$, and the double quartic curve \mathcal{C}_i belongs to $|\mathcal{O}_{H_i \cap Y}(2)|$. In the above homogeneous coordinates the intersection $H_3 \cap H_4$ is a plane defined by $z_3 = z_4 = 0$, and $H_3 \cap H_4 \cap Y$ is a conic defined by $z_0^2 = z_1z_2$, on which the 4 points $\mathcal{C}_3 \cap \mathcal{C}_4$ lie. Conics on the plane passing through these 4 points form a pencil, which is invariant under the real structure. Choose any real one of such conics, and let $q(z_0, z_1, z_2)$ be its defining equation with real coefficients, which is of course uniquely determined up to rescaling. Among the above 26 points there are exactly 8 points lying on H_3 (which are the points 3, 4 and 7 to 12 in Figure 4), four of which are the above 4 points on $H_3 \cap H_4$ (the points 9, 10, 11, 12 in Figure 4). Any quadratic polynomial on H_3 whose restriction to $H_3 \cap H_4$ equals q is of the form

$$(4.8) \quad Q_3 = q(z_0, z_1, z_2) + a_0z_0z_3 + a_1z_1z_3 + a_2z_2z_3 + a_3z_3^2.$$

Imposing that the quadric (Q_3) passes the remaining 4 points (3, 4, 7, 8), the coefficients a_0, a_1, a_2 and a_3 are uniquely determined (without an ambiguity of rescaling), and they are real since the set of 4 points (3, 4, 7, 8) is real. Then since elements of $|\mathcal{O}_{H_3 \cap Y}(2)|$ which go through the 8 points is unique by dimension counting, it follows that the quadratic surface (Q_3) automatically contains \mathcal{C}_3 . The situation is the same for H_4 , and let

$$(4.9) \quad Q_4 = q(z_0, z_1, z_2) + b_0z_0z_4 + b_1z_1z_4 + b_2z_2z_4 + b_3z_4^2.$$

be the quadratic polynomial on H_4 with real coefficients, which is uniquely determined by q and the 8 points (1, 2, 5, 6 and 9 to 12) on H_4 . Then $(Q_4) \supset \mathcal{C}_4$.

Since $Q_3|_{H_3 \cap H_4} = Q_4|_{H_3 \cap H_4} (= q)$, the pair (Q_3, Q_4) is naturally regarded as a section of a line bundle $\mathcal{O}_{H_3 \cup H_4}(2)$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^4} \xrightarrow{\otimes z_3z_4} \mathcal{O}_{\mathbb{CP}^4}(2) \longrightarrow \mathcal{O}_{H_3 \cup H_4}(2) \longrightarrow 0,$$

we obtain that the section (Q_3, Q_4) can be extended to a quadratic polynomial on \mathbb{CP}^4 , and that such polynomial is unique up to adding a constant multiple of z_3z_4 . Explicitly such an extension has to be of the form

$$(4.10) \quad Q(z_0, z_1, z_2, z_3, z_4) := q(z_0, z_1, z_2) + \sum_{0 \leq i \leq 3} a_i z_i z_3 + \sum_{0 \leq i \leq 3} b_i z_i z_4 + c z_3 z_4,$$

where c is an arbitrary constant. Of course, the hyperquadric satisfies $(Q) \supset \mathcal{C}_3 \cup \mathcal{C}_4$. Then if we further impose that (Q) contains the point $p_2 \in \mathcal{C}_1 \cap \mathcal{C}_2 \subset l = \{z_0 = z_1 = z_2 = 0\}$, then from (4.10) a linear equation for a_3, b_3, c is obtained, from which c is uniquely determined. c is real since a_3 and b_3 are real. Summarizing up, we have obtained that once we fix a real quadratic polynomial $q(z_0, z_1, z_2)$, then there exists a unique real quadratic polynomial $Q(z_0, z_1, z_2, z_3, z_4)$ whose restriction to $H_3 \cap H_4$ equals q and which goes through the 9 points (1 to 8 and p_2). In particular, if $q = z_0^2 - z_1 z_2$, then $Q = z_0^2 - z_1 z_2$.

Next we show that this polynomial Q (which is uniquely determined from q) always contains the double curves \mathcal{C}_i for any $1 \leq i \leq 5$. It remains to show $\mathcal{C}_i \subset (Q)$ for $i = 1, 2, 5$. For $i = 1, 2$ this is immediate since (Q) already goes through 5 points on the conic \mathcal{C}_i , which means that it goes through the remaining 1 point \bar{p}_2 . $\mathcal{C}_5 \subset (Q)$ is also immediate if we notice that as $\mathcal{C}_i \subset (Q)$ for $1 \leq i \leq 4$, (Q) already goes through all the 12 points $\mathcal{C}_i \cap \mathcal{C}_5$ for $1 \leq i \leq 4$ on H_5 among the 26 points obtained in Section 4.2 and that since $h^0(\mathcal{O}_{H_5 \cap Y}(2)) = 9$, eight points already and uniquely determine the quadric. Thus we have proved the existence of a quadratic polynomial Q satisfying $\mathcal{C}_i \subset (Q)$ for any $1 \leq i \leq 5$.

For the uniqueness in the sense of the proposition, let Q and Q' be as stated in the proposition. Then since both $Q|_{H_3 \cap H_4}$ and $Q'|_{H_3 \cap H_4}$ are real and belong to the pencil (determined from the 4 points $\mathcal{C}_3 \cap \mathcal{C}_4$), there exists $(c, c') \in \mathbb{R}^2$ with $(c, c') \neq (0, 0)$ such that $cQ - c'Q'|_{H_3 \cap H_4} \in (z_0^2 - z_1 z_2)$. Further, the hyperquadric $(cQ - c'Q')$ goes through the points 1 to 8 and p_2 at least, and therefore must belong to the ideal $(z_0^2 - z_1 z_2)$ by the uniqueness which was already proved. Thus we get the required uniqueness. \square

4.3. Defining equation of the branch divisor. With the results in the previous 2 subsections, we are ready to provide the main result in this paper:

Theorem 4.5. *Let Z be any twistor space on $4\mathbb{CP}^2$ containing the surface S (constructed in Section 2) as a real member of $|F|$. Let $\Phi_4 : Z_4 \rightarrow Y$ be the generically 2 to 1 morphism canonically obtained from the explicit birational transformations in Section 3, and B the branch divisor of Φ_4 . Let $Q(z_0, z_1, z_2, z_3, z_4)$ be a defining equation of the hyperquadric containing all the 5 double curves, obtained in Proposition 4.4. Then B is an intersection of the scroll $Y = \{z_0^2 = z_1 z_2\}$ with the quartic hypersurface defined by the equation of the form*

$$(4.11) \quad z_0 z_3 z_4 f(z_0, z_1, z_2, z_3, z_4) = Q(z_0, z_1, z_2, z_3, z_4)^2$$

where $f(z_0, z_1, z_2, z_3, z_4)$ is a linear polynomial with real coefficients.

Proof. By Proposition 3.4 there exists a real hyperquartic such that the intersection with Y is the branch divisor B . Let $\mathcal{B} \subset \mathbb{CP}^4$ be any one of such hyperquartics and $F = F(z_0, z_1, z_2, z_3, z_4)$ a defining equation of \mathcal{B} . We note that \mathcal{B} is not unique in the sense that F is determined only up to quartic polynomials in the ideal $(z_0^2 - z_1 z_2)$. Then for $i = 1, 2$, the restriction of \mathcal{B} to a plane $H_i \cap Y = \{z_0 = z_i = 0\}$ is the twice of the double conic \mathcal{C}_i (see (4.7)). Also, by the choice of Q , the restriction of the hyperquadric (Q) to the same plane is \mathcal{C}_i . These two mean that there exists a constant $c_i \in \mathbb{C}$ such that $F - c_i Q^2$ belongs to the ideal (z_0, z_i) . Namely there exist cubic polynomials f_i and g_i (in z_0, z_1, z_2, z_3, z_4) satisfying

$$(4.12) \quad F - c_i Q^2 = z_0 f_i + z_i g_i \quad (i = 1, 2).$$

Taking the difference, we obtain

$$(4.13) \quad (c_1 - c_2) Q^2 = z_0(f_1 - f_2) + z_1 g_1 - z_2 g_2.$$

If $c_1 \neq c_2$, substituting $z_0 = z_1 = 0$, the hyperquadric (Q) restricted to the plane $\{z_0 = z_1 = 0\}$ is defined by $z_2g_2 = 0$. This means that \mathcal{C}_1 is reducible, which cannot happen since it is the image of the twistor line L_1 (see (4.1)). Hence we obtain $c_1 = c_2$. Similarly, for $i = 3, 4$, considering the restrictions of F and Q^2 to the cone $H_i \cap Y = \{z_i = z_0^2 - z_1z_2 = 0\}$, again by coincidence, there exist a constant $c_i \in \mathbb{C}$, a cubic polynomial f_i and a quadratic polynomial g_i satisfying

$$(4.14) \quad F - c_i Q^2 = z_i f_i + (z_0^2 - z_1 z_2) g_i \quad (i = 3, 4).$$

By (4.12) with $i = 1$ and (4.14) we obtain

$$(4.15) \quad (c_3 - c_1) Q^2 = z_0 f_1 + z_1 g_1 - z_i f_i - (z_0^2 - z_1 z_2) g_i \quad (i = 3, 4).$$

From this we again obtain that the hyperquadric $\{Q = 0\}$ restricted to the plane $\{z_0 = z_1 = 0\}$ is given by $\{z_i f_i = 0\}$, contradicting the irreducibility of \mathcal{C}_1 . Hence we obtain $c_1 = c_i$ for $i = 3, 4$. Thus we get $c_1 = c_2 = c_3 = c_4$. If $c_1 = 0$, by (4.12), we have $F = z_0 f_1 + z_1 g_1$. But this cannot happen since this means $\mathcal{B} \supset \{z_0 = z_1 = 0\}$, contradicting $\mathcal{B} \cap \{z_0 = z_1 = 0\} = \mathcal{C}_1$. Hence $c_1 \neq 0$. So replacing Q with $Q/\sqrt{c_i}$, we may assume that all the four c_i -s in (4.12) and (4.14) are one.

Next in the expression (4.12) we take f_i and g_i in such a way that g_i does not contain z_0 . Then since the right hand side of (4.13) is zero, $z_1 g_1 - z_2 g_2 = 0$ follows. Hence $g_1 \in (z_2)$, and we can write $g_1 = z_2 h_1$ by a quadratic polynomial h_1 which does not contain z_0 . Similarly, in the expression (4.14) we take f_i and g_i in such a way that f_3 and f_4 do not belong to the ideal $(z_0^2 - z_1 z_2)$. Then this time from (4.14) for the case $i = 3$ and $i = 4$, we obtain

$$(4.16) \quad (z_3 f_3 - z_4 f_4) + (z_0^2 - z_1 z_2)(g_3 - g_4) = 0.$$

From the choice of f_3 and f_4 , it follows $f_3 \in (z_4)$ and $f_4 \in (z_3)$. Hence we can put $f_3 = z_4 f_5$ for some quadratic polynomial f_5 . From these, we obtain

$$(4.17) \quad F - Q^2 = z_0 f_1 + z_1 z_2 h_1 = z_3 z_4 f_5 + (z_0^2 - z_1 z_2) g_3.$$

Then since h_1 does not contain z_0 , from the latter equality we can readily deduce that if we write $f_5 = z_0 f_6 + f_7$ in a way that f_7 does not contain z_0 , then f_7 is a multiple of $z_1 z_2$, so that $f_5 = z_0 f_6 + c z_1 z_2$ for some $c \in \mathbb{C}$. Hence by (4.17) we obtain

$$(4.18) \quad F - Q^2 = z_3 z_4 (z_0 f_6 + c z_1 z_2) + (z_0^2 - z_1 z_2) g_3.$$

Defining a linear polynomial f_8 by $f_6 = -c z_0 + f_8$ (so that f_8 may contain z_0) and substituting into (4.18), we finally get

$$(4.19) \quad F - Q^2 = z_0 z_3 z_4 f_8 + (z_0^2 - z_1 z_2)(g_3 - c z_3 z_4).$$

Thus we obtain $F = Q^2 + z_0 z_3 z_4 f_8 + (z_0^2 - z_1 z_2)(g_3 - c z_3 z_4)$. Hence modulo quartic polynomials in the ideal $(z_0^2 - z_1 z_2)$, \mathcal{B} is defined by the equation of the form (4.11). This completes a proof of the theorem. \square

From the quartic equation (4.11) it is immediate to see that the intersection of the hyperquadric (Q) and the hyperplanes H_3 and H_4 are double quadric surfaces, and this is of course consistent with the fact that the restrictions $B|_{H_3 \cap Y}$ and $B|_{H_4 \cap Y}$ are double curves. On the other hand, for $i = 1, 2$, in order to see that $B|_{H_i \cap Y}$ are also double curves from the equation, we just need to notice that, on the scroll Y , $z_i = 0$ means $z_0 = 0$.

In comparison with the case of $3\mathbb{CP}^2$, appearance of the linear polynomial f in our defining equation (4.11) might look strange at first sight. As the following proposition shows, f comes from the *fifth* double curve \mathcal{C}_5 , which does not exist in the case of $3\mathbb{CP}^2$:

Proposition 4.6. *Up to non-zero constants, the linear polynomial f in (4.11) is exactly z_5 we have defined in (4.6). In other words, for the third double quartic curve, we have $\mathcal{C}_5 = \{f = Q = z_0^2 - z_1z_2 = 0\}$.*

Proof. We will find all hyperplanes in \mathbb{CP}^4 (which is the target space of the anticanonical map Φ), which correspond to reducible members of the anticanonical system $|2F|$. First as above for any hyperplane H defined by the equation of the form $a_0z_0 + a_1z_1 + a_2z_2 = 0$, the corresponding member $\Phi^{-1}(H) \in |2F|$ is clearly reducible. (We are including a non-reduced case.) Also, $\Phi^{-1}(H_3)$ and $\Phi^{-1}(H_4)$ are reducible since $B \cap H_3$ and $B \cap H_4$ are double curves. By the same reason, if $H_f = \{f = 0\}$, the divisor $\Phi^{-1}(H_f)$ is reducible. Then recalling that the double quartic curve \mathcal{C}_5 is obtained as an image of the third reducible member of $|2F|$ obtained in Proposition 4.1, in order to prove the claim of the proposition, it suffices to show that there exists no other hyperplane H such that $\Phi^{-1}(H)$ is reducible.

If H is such a hyperplane, then either $Y \cap H$ is reducible, or $Y \cap H$ is irreducible (i.e. a cone) and $B|_{Y \cap H}$ is a double curve. The former occurs exactly when H is defined by the equation of the form $a_0z_0 + a_1z_1 + a_2z_2 = 0$. So suppose the latter happens. Then recalling $B = \mathcal{B} \cap Y$ and $H|_Y$ is reduced by the assumption, $B|_{Y \cap H}$ can be a double curve only when $\mathcal{B}|_H$ is a double surface. We show by algebraic mean that this happens only when H is defined by one of the 4 factors of the left-hand side of (4.11).

If $\mathcal{B}|_H$ is a double surface, there exists a quadratic polynomial q on H such that $(z_0z_3z_4f - Q^2)|_H = q^2$; namely

$$(4.20) \quad z_0z_3z_4f|_H = (Q|_H)^2 + q^2.$$

If H is defined by the equation of the form $z_1 = b_0z_0 + b_2z_2 + b_3z_3 + b_4z_4$, then even after substitution the left-hand side of (4.20) does not have monomial of the form $z_i^3z_j$ for any $1 \leq i, j \leq 4$. Therefore the coefficient of $z_i^3z_j$ of the right-hand side of (4.20) must be zero for any $1 \leq i, j \leq 4$. By an elementary argument, it is possible to show this can happen only when $(Q|_H)^2 + q^2 = 0$. Hence by (4.20) H is equal to one of H_0, H_3, H_4 and H_f . By symmetry of the equation, if H is defined by a equation of the form $z_2 = b_0z_0 + b_1z_2 + b_3z_3 + b_4z_4$, then (4.20) is possible only when H is one of H_0, H_3, H_4 and H_f . If H is of the form $z_0 = b_3z_3 + b_4z_4$, then the left-hand side of (4.20) cannot contain monomials of the form $z_i^4, z_1^3z_i, z_2^3z_i$ (for any i), and $z_1^2z_3z_4, z_1z_2z_3z_4, z_2^2z_3z_4$. Therefore the coefficients of these monomials of the right-hand side of (4.20) must vanish. From these, again by an elementary argument it is possible to show that $(Q|_H)^2 + q^2 = 0$. Hence again H has to be one of H_0, H_3, H_4 and H_f . The remaining 2 cases immediately follow from symmetry of the equation. Thus we have shown that $\mathcal{B}|_H$ is a double surface only when H is one of H_0, H_3, H_4 and H_f . \square

We again emphasize that the role of the 3 double quartic curves is symmetric, and any choice of two leads to the equation of the form (4.11).

Remark 4.7. One may wonder whether the linear polynomial f can be taken as one of the homogenous coordinates on \mathbb{CP}^4 . At least in generic situation this is possible, but if we do so, we lose simplicity of the defining equation of the scroll Y , and it makes more difficult the counting the number of effective parameters in defining equations of $\mathcal{B} \cap Y$ which will be done in Section 5.1.

4.4. The number of singularities of the branch divisor. In this subsection by using the quartic equation obtained in the previous subsection we determine the number of singularities of the branch divisor of the double covering. Similarly to the method by Kreussler

[10] and Kreussler-Kurke [12], we resort to *topology*; more precisely we compute the Euler number of the relevant spaces to determine the number of singularities. Though we require much more complicated computation than the case of $3\mathbb{CP}^2$, we do it since this result is crucial for determining the dimension of the moduli space of the present twistor spaces.

As in the proof of Proposition 3.7 let $Z_4 \xrightarrow{\mu_5} Z_5 \xrightarrow{\Phi_5} Y$ be the Stein factorization of the degree 2 morphism $\Phi_4 : Z_4 \rightarrow Y$. We already know that μ_5 contracts finitely many curves, whose images are singular points of the branch divisor B . If we put $l_5 = \mu_5(l_4)$ and $\bar{l}_5 = \mu_5(\bar{l}_4)$, Z_5 also has ordinary double points along $l_5 \cup \bar{l}_5$. All other singularities of Z_5 are lying on singularities of the branch divisor B . Among these singularities we already know that there are 26 singularities listed in Section 4.2, and 24 points among them are ordinary double points. The 2 points excluded here are exactly the points $\mathcal{C}_1 \cap \mathcal{C}_2$, for which we still denote by p_2 and \bar{p}_2 (see Figure 4 again). We begin with determining the type of singularities of these 2 points:

Proposition 4.8. *At the 2 points p_2 and \bar{p}_2 , the branch divisor B has A_3 -singularities.*

Proof. Recall that $l = \{z_0 = z_1 = z_2 = 0\}$, and $\mathcal{C}_1 \cap \mathcal{C}_2 = B \cap l = \{Q = 0\} \cap l$. As above let p_2 be any one of the 2 points and we work in a neighborhood of p_2 . We put $x := z_0/z_4, y := z_1/z_4, z := z_2/z_4$ and $u := Q$. Then by transversality for the intersection of Q and l , we can use (x, y, z, u) as coordinates in a neighborhood of p_2 in \mathbb{CP}^4 , and noticing $z_3 z_4 f \neq 0$ at p_2 , we may suppose that the hyperquartic (4.11) is defined by a very simple equation, $x = u^2$. Since Y is defined by $x^2 = yz$, we deduce that p_2 is an A_3 -singular point of the surface B . By reality, \bar{p}_2 is also an A_3 -singular point. \square

Next, as the transformation from Z to Z_4 is explicit, it is easy to show the following:

Proposition 4.9. *For the variety Z_4 we have $e(Z_4) = 10$.*

Proof. Since Z is a twistor space on $4\mathbb{CP}^2$, we have $e(Z) = 2 + 2(b_2(4\mathbb{CP}^2) + 1) = 12$. Because the blowup μ_1 replaces two disjoint \mathbb{CP}^1 -s by two $\mathbb{CP}^1 \times \mathbb{CP}^1$ -s, we have $e(Z_1) = 12 + 4 = 16$. Then since a flop does not change the Euler number we obtain $e(Z_3) = 16$. Finally looking Figure 3 (d), the exceptional divisor $E_1 \cup \bar{E}_1$ of the contraction $\mu_4 : Z_3 \rightarrow Z_4$ has Euler number 8, and the image $l_4 \cup \bar{l}_4$ of the exceptional divisor has Euler number 2. Hence we obtain $e(Z_4) = 16 - (8 - 2) = 10$. \square

The next result means that the 26 points that we have already found are *not* all singularities of the branch divisor B , but in the generic situation B has extra 6 ordinary double points:

Theorem 4.10. *Let $\{b_1, \dots, b_k\}$ be the set of all singular points of B which are different from the 26 singular points listed in Section 4.2. Let μ_i be the Milnor number of the singular point b_i , and put $\beta_i := \mu_5^{-1}(b_i)$ for the exceptional curve of μ_5 over the point b_i . (Of course we do not assume irreducibility of β_i .) Then we have the relation*

$$(4.21) \quad \sum_{i=1}^k \{e(\beta_i) + \mu_i - 1\} = 12.$$

In particular, if all the singularities are ordinary double points, we have $k = 6$.

Proof. First since $\mu_5 : Z_4 \rightarrow Z_5$ replaces each of the 24 ordinary double points by smooth \mathbb{CP}^1 and also replaces the singular point b_i by the curve β_i for $1 \leq i \leq k$, we obtain

$$(4.22) \quad e(Z_4) = e(Z_5) + 24 + \sum_{1 \leq i \leq k} \{e(\beta_i) - 1\}.$$

On the other hand by the double covering $Z_5 \rightarrow Y$ we have $e(Z_5) = 2e(Y) - e(B)$. Further as Y is obtained from the \mathbb{CP}^2 -bundle \tilde{Y} over \mathbb{CP}^1 , and it replaces $\mathbb{CP}^1 \times \mathbb{CP}^1$ by $l \simeq \mathbb{CP}^1$, we have $e(Y) = e(\tilde{Y}) - (4 - 2) = 6 - 2 = 4$. Hence we have $e(Z_5) = 8 - e(B)$, giving

$$(4.23) \quad e(Z_4) = 32 - e(B) + \sum_{1 \leq i \leq k} \{e(\beta_i) - 1\}.$$

Next for computing $e(B)$ let D be a general member of the system $|\mathcal{O}_Y(4)|$. Then since the scroll Y has ordinary double points along the line l , the divisor D has ordinary double points at the 4 points $D \cap l$. As before let $\nu : \tilde{Y} \rightarrow Y$ be the blowup at l , and let \tilde{D} be the strict transform of D . By Bertini's theorem we may suppose that \tilde{D} is non-singular. We shall compute $e(\tilde{D})$.

As before write $\mathfrak{f} := \tilde{\pi}^* \mathcal{O}_\Lambda(1) \in H^2(\tilde{Y}, \mathbb{Z})$. From the standard relationship of the total Chern class $c(T_{\tilde{Y}}|_{\tilde{D}}) = c(T_{\tilde{D}}) \cdot c(N_{\tilde{D}/\tilde{Y}})$ and adjunction formula, we readily obtain

$$(4.24) \quad e(\tilde{D}) = c_2(T_{\tilde{D}}) = c_2(T_{\tilde{Y}}) \cdot \tilde{D} + (K_{\tilde{Y}} + \tilde{D}) \cdot \tilde{D} \cdot \tilde{D},$$

where the dot means the product in $H^*(\tilde{Y}, \mathbb{Z})$. As in the proof of Proposition 3.4 let $\mathcal{O}(0, 1) := (\nu^* \mathcal{O}_l(1))|_\Sigma$, so that $\mathcal{O}(1, 0) = \mathfrak{f}|_\Sigma$. Then by using $N_{\Sigma/\tilde{Y}} \simeq \mathcal{O}(-2, 1)$ and the adjunction formula applied to a fiber of $\tilde{\pi}$, we readily obtain $K_{\tilde{Y}} \sim -3\Sigma - 6\mathfrak{f}$. Further, in the cohomology ring of \tilde{Y} , we have $\Sigma^3 = N_{\Sigma/\tilde{Y}} \cdot N_{\Sigma/\tilde{Y}} = -4$, $\Sigma^2 \cdot \mathfrak{f} = N_{\Sigma/\tilde{Y}} \cdot \mathfrak{f} = \mathcal{O}(-2, 1) \cdot \mathcal{O}(1, 0) = 1$, $\Sigma \cdot \mathfrak{f}^2 = 0$ and $\mathfrak{f}^3 = 0$. Furthermore by recalling $\nu^* \mathcal{O}(1) \sim \Sigma + 2\mathfrak{f}$ (see (3.6)), we obtain $K_{\tilde{Y}} + \tilde{D} \sim (-3\Sigma - 6\mathfrak{f}) + 4(\Sigma + 2\mathfrak{f}) = \Sigma + 2\mathfrak{f}$. From these we readily get $(K_{\tilde{Y}} + \tilde{D}) \cdot \tilde{D} \cdot \tilde{D} = 32$.

For computing $c_2(T_{\tilde{Y}}) \in H^4(\tilde{Y}, \mathbb{Z})$, as generators of $H^4(\tilde{Y}, \mathbb{Z})$ we take any element $\zeta \in |\mathcal{O}_\Sigma(1, 0)|$ and $\eta \in |\mathcal{O}_\Sigma(0, 1)|$, viewed as submanifolds in \tilde{Y} , and put $c_2(T_{\tilde{Y}}) = a\zeta + b\eta$. From the exact sequence associated to the inclusion $\Sigma \subset \tilde{Y}$, we immediately obtain $c_2(T_{\tilde{Y}})|_\Sigma = c_1(\Sigma) \cdot c_1(N_{\Sigma/\tilde{Y}}) + c_2(\Sigma)$. Then since $c_1(\Sigma) = \mathcal{O}(2, 2)$, $c_1(N_{\Sigma/\tilde{Y}}) = (-2, 1)$ and $c_2(\Sigma) = e(\Sigma) = 4$, we obtain $c_2(T_{\tilde{Y}})|_\Sigma = 2$. On the other hand, from the inclusion $\mathfrak{f} \subset \tilde{Y}$ we readily obtain $c_2(T_{\tilde{Y}})|_{\mathfrak{f}} = 3$. Further, in the cohomology ring of \tilde{Y} we have $\zeta \cdot \Sigma = \mathcal{O}(1, 0) \cdot \mathcal{O}(-2, 1) = 1$, $\eta \cdot \Sigma = \mathcal{O}(0, 1) \cdot \mathcal{O}(-2, 1) = -2$, $\zeta \cdot \mathfrak{f} = \mathcal{O}(1, 0) \cdot \mathcal{O}(1, 0) = 0$ and $\eta \cdot \mathfrak{f} = \mathcal{O}(0, 1) \cdot \mathcal{O}(1, 0) = 1$. Therefore by restricting to Σ and \mathfrak{f} respectively, we get $a - 2b = 2$ and $b = 3$. Hence $a = 8$. Therefore we obtain $c_2(\tilde{Y}) \cdot \tilde{D} = (8\zeta + 3\eta) \cdot 4(\Sigma + 2\mathfrak{f})$, which is readily computed to be 32. Hence from (4.24) we obtain $e(\tilde{D}) = 32 + 32 = 64$.

As $\tilde{D} \rightarrow D$ contracts four \mathbb{CP}^1 -s to 4 points, we have $e(D) = 60$. Then D is obtained from the actual branch divisor B by (a) smoothing the 24 nodes, (b) smoothing k singular points b_1, \dots, b_k , and (c) deforming each of the two A_3 -singularities (which is exactly $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p_2, \bar{p}_2\}$) to two A_1 singularities. Adding the Milnor number of the singularities for the cases (a) and (b), and also taking the difference of the Milnor number of A_3 -singularity and two A_1 -singularities into account, we obtain

$$(4.25) \quad e(B) = e(D) - 26 - \sum_{1 \leq i \leq k} \mu_i,$$

and hence $e(B) = 34 - \sum_{1 \leq i \leq k} \mu_i$. Substituting this into (4.23) and using Proposition 4.9, we obtain (4.21). \square

Remark 4.11. If the blown-up 8 points on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are arranged as in Figure 1 is in a general position (in certain precise sense), then C_2 and \overline{C}_2 are all curves that are contracted to points by the bi-anticanonical map. But if the 8 points are in a special position (in certain precise sense), then the map contracts extra curves. It is not difficult to classify all positions which yield this situation. The appearance of this kind of curves is exactly the reason why the anticanonical map of the twistor spaces contracts some rational curves which cannot be found from the equation of the branch divisor.

5. MODULI SPACE AND GENERICITY OF THE TWISTOR SPACES

5.1. Dimension of the moduli space. In this subsection we compute the dimension of the moduli space of our twistor spaces by counting the number of effective parameters, and verify that it agrees with the dimension of the cohomology group which is relevant to the present case.

For the former purpose, we recall from Section 3 that Z canonically determines a birational model Z_4 and the degree 2 morphism $\Phi_4 : Z_4 \rightarrow Y$, and from Section 4.3 the branch divisor of Φ_4 is an intersection of Y with the quartic surface defined by

$$(5.1) \quad z_0 z_3 z_4 f(z_0, z_1, z_2, z_3, z_4) = Q(z_0, z_1, z_2, z_3, z_4)^2,$$

where f and Q are linear and quadratic polynomials with real coefficients respectively. In this subsection we denote this quartic hypersurface by $\mathcal{B}(f, Q)$. Since the quartic hypersurface $\mathcal{B}(f, Q)$ uniquely determines the double cover via the natural quadratic map $\mathcal{O}_Y(2) \rightarrow \mathcal{O}_Y(4)$ (which takes squares), up to small resolutions, Z is uniquely determined by the quartic hypersurface. Of course, f has 5 coefficients and Q has 15 coefficients, so the equation (5.1) contains 20 parameters. Further it is elementary to see that two pairs (f, Q) and (f', Q') of linear and quadratic polynomials (over \mathbb{R}) determine the same hyperquartic surface if and only if $(f', Q') = (c^2 f, cQ)$ for some $c \in \mathbb{R}$. This decreases the number of parameters by one. On the other hand, projective transformations which preserve Y and which preserves *the form of* the equation (5.1) have to be $(z_0, z_1, z_2, z_3, z_4) \mapsto (abz_0, az_1, bz_2, cz_3, dz_4)$ for some $a, b, c, d \in \mathbb{R}^\times$. (Here we are only considering transformations which are homotopic to the identity.) If a pair (f', Q') is obtained from a pair (f, Q) by one of these transformations, then the intersections $Y \cap \mathcal{B}(f, Q)$ and $Y \cap \mathcal{B}(f', Q')$ are mutually biholomorphic, so that they define mutually isomorphic double cover. But taking an effect of the above equivalence $(c^2 f, cQ) = (f, Q)$ into account, we can suppose $ab = 1$ and therefore these projective transformations decrease the number of parameters by 3. Thus up to now the number of parameters is $20 - (1 + 3) = 16$. However what we have to consider is not the hyperquartics (5.1) themselves but the intersection with Y ; namely if $Q' = Q + c(z_0^2 - z_1 z_2)$ for some $c \in \mathbb{R}$, then we have the coincidence $Y \cap \mathcal{B}(f, Q) = Y \cap \mathcal{B}(f, Q') \subset \mathbb{CP}^4$. Clearly these transformations are not included in the above projective transformations, so they drop the dimension by one. Thus we have obtained that the number of effective parameters in the quadratic hypersurface (5.1) is 15. Finally, by Theorem 4.10, the pair (f, Q) must satisfy the constraint that $\mathcal{B}(f, Q) \cap Y$ has extra 6 ordinary double points in general, which decreases the number of parameters by 6. Therefore we conclude that *the space of isomorphic classes of the divisors of the form $\mathcal{B}(f, Q) \cap Y$, which can be the branch divisor for the twistor spaces under consideration, is 15-dimensional.*

Next we compute the dimension of the moduli space of our twistor spaces by determining the dimension of the first cohomology group of an appropriate subsheaf of the tangent sheaf. We begin with a computation for the full moduli space.

Proposition 5.1. *Let Z be a twistor space on $4\mathbb{CP}^2$ which contains the surface S constructed in Section 2 as a real member of $|F|$. Then we have $H^i(Z, \Theta_Z) = 0$ for $i \neq 1$ and $h^1(Z, \Theta_Z) = 13$.*

Proof. As computed in [14], for any twistor space on $n\mathbb{CP}^2$, by the Riemann-Roch formula, we have $\chi(\Theta_Z) = 15 - 7n$. Also, since Z is Moishezon and $|F|$ has an irreducible member, we have $H^2(\Theta_Z) = 0$ by [2]. Further we always have $H^3(\Theta_Z) = 0$. Hence it suffices to show $H^0(\Theta_Z) = 0$. Let $\text{Aut}_0 Z$ be the identity component of the holomorphic automorphism group of Z . Then the real part $(\text{Aut}_0 Z)^\sigma$ is naturally identified with the identity component of conformal automorphism group of the self-dual structure. Also, since $|F|$ has just 2 irreducible components, $(\text{Aut}_0 Z)^\sigma$ acts on $S_1^+ \cup S_1^-$. Hence as the twistor projections $S_1^+ \rightarrow 4\mathbb{CP}^2$ and $S_1^- \rightarrow 4\mathbb{CP}^2$ are of degree 1, $(\text{Aut}_0 Z)^\sigma$ acts effectively on $S_1^+ \cup S_1^-$. Furthermore, the degree 1 divisor S_1^+ and S_1^- are obtained from \mathbb{CP}^2 by blowing-up 4 points, exactly 3 of which are collinear. From this it readily follows that the subgroup of $\text{Aut } S_0^+$ which consists of automorphism preserving the twistor line L_1 is 0-dimensional. Hence so is $\text{Aut}(S_0^+ \cup S_0^-)$. Thus $\text{Aut}_0 Z$ cannot be of positive dimension. Therefore $H^0(\Theta_Z) = 0$. \square

From the proposition, the real part of the Kuranishi family of our twistor space Z is 13-dimensional. Of course, generic twistor spaces on $4\mathbb{CP}^2$ is algebraic dimension 1 and generic members of the Kuranishi family have the same property. In order to restrict to the Moishezon twistor spaces under consideration, we show the following.

Proposition 5.2. *Let Z and S be as in Proposition 5.1 and C_1 and \overline{C}_1 the (-3) -curves on S . Then deformation theory of the pair $(Z, C_1 \sqcup \overline{C}_1)$ is unobstructed and its Kuranishi family is 9-dimensional. Further, for all sufficiently small deformations preserving the real structure in the Kuranishi family, the twistor spaces contain a non-singular surface constructed in Section 2 as a real member of $|F|$.*

Of course, the last property means that the deformed spaces are still the Moishezon twistor spaces under consideration.

Proof of Proposition 5.2. Let $\Theta_{Z, C_1 + \overline{C}_1}$ be the subsheaf of Θ_Z whose germs are vector fields that are tangents to C_1 and \overline{C}_1 . For the former claim on the Kuranishi family, it suffices to show that $H^2(Z, \Theta_{Z, C_1 + \overline{C}_1}) = 0$ and $h^1(Z, \Theta_{Z, C_1 + \overline{C}_1}) = 9$. Recalling $N_{C_1/Z} \simeq N_{\overline{C}_1/Z} \simeq \mathcal{O}(-2)^{\oplus 2}$, we obtain the standard exact sequence

$$0 \longrightarrow \Theta_{Z, C_1 + \overline{C}_1} \longrightarrow \Theta_Z \longrightarrow \mathcal{O}_{C_1}(-2)^{\oplus 2} \oplus \mathcal{O}_{\overline{C}_1}(-2)^{\oplus 2} \longrightarrow 0,$$

which induces an exact sequence

$$(5.2) \quad 0 \longrightarrow H^1(\Theta_{Z, C_1 + \overline{C}_1}) \longrightarrow H^1(\Theta_Z) \longrightarrow \mathbb{C}^4 \longrightarrow H^2(\Theta_{Z, C_1 + \overline{C}_1}) \longrightarrow 0.$$

Hence with the aid of Proposition 5.1 we have only to show $H^2(\Theta_{Z, C_1 + \overline{C}_1}) = 0$. For this we first deduce from duality and rationality that $H^2(\Theta_S(-C_1 - \overline{C}_1)) = 0$, which implies, from the exact sequence $0 \rightarrow \Theta_S(-C_1 - \overline{C}_1) \rightarrow \Theta_{S, C_1 + \overline{C}_1} \rightarrow \Theta_{C_1 \sqcup \overline{C}_1} \rightarrow 0$, that $H^2(\Theta_{S, C_1 + \overline{C}_1}) = 0$. Moreover, noting $N_{S/Z} \simeq F|_S \simeq -K_S$, we have an exact sequence

$$0 \longrightarrow \Theta_{S, C_1 + \overline{C}_1} \longrightarrow \Theta_{Z, C_1 + \overline{C}_1}|_S \longrightarrow -K_S \otimes \mathcal{O}_S(-C_1 - \overline{C}_1) \longrightarrow 0.$$

For the last term we have $-K_S \otimes \mathcal{O}_S(-C_1 - \overline{C}_1) \simeq \mathcal{O}_S(C_2 + \overline{C}_2)$, and it is easy to see $H^2(\mathcal{O}_S(C_2 + \overline{C}_2)) = 0$. Hence for the middle term we obtain $H^2(\Theta_{Z, C_1 + \overline{C}_1}|_S) = 0$. Then by the exact sequence $0 \rightarrow \Theta_Z(-S) \rightarrow \Theta_{Z, C_1 + \overline{C}_1} \rightarrow \Theta_{Z, C_1 + \overline{C}_1}|_S \rightarrow 0$ and $H^2(\Theta_Z(-S)) = 0$ [2], we finally obtain $H^2(\Theta_{Z, C_1 + \overline{C}_1}) = 0$, as claimed.

For the latter claim about the existence of the surface in the deformed space, let Z_t be any one of the deformed twistor space which is sufficiently close to the original Z , and $C_{1t}, \overline{C}_{1t} \subset Z_t$ the curves corresponding to the original curves C_1 and \overline{C}_1 . Let F_t be the fundamental line bundle on Z_t . Then as $\dim |F| = 1$, we may suppose $\dim |F_t| = 1$ by upper-semicontinuity of dimensions of cohomology groups under deformations and the Riemann-Roch formula $\chi(F_t) = 2$. We also have an invariance $F_t \cdot C_{1t} = F \cdot C_1$, and the latter is equal to $K_S^{-1} \cdot C_1 = -1$, and therefore $F_t \cdot C_{1t} = -1$. This means that C_{1t} and \overline{C}_{1t} are base curves of the pencil $|F_t|$. Let $S_t \in |F|$ be any real irreducible member. Through the Kuranishi family, this surface can be regarded as a small deformation of some real irreducible $S \in |F|$, which means that S_t is obtained from $\mathbb{CP}^1 \times \mathbb{CP}^1$ by moving the blowup 8 points from the original positions (indicated as in Figure 1). But since S_t contains the curves C_{1t} and \overline{C}_{1t} as (-3) -curves, the property that 3 points belong to a $(1, 0)$ -curve must be preserved. This means that the structure of S_t is the same as that of the original S , and we are done. \square

5.2. Genericity of the twistor spaces. In this subsection, by using a theorem of Pedersen-Poon about structure of real irreducible members of $|F|$, we show that the present twistor spaces are in a sense generic among all Moishezon twistor spaces on $4\mathbb{CP}^2$. We first recall the theorem of Pedersen-Poon [15] in a precise form:

Proposition 5.3. *Let Z be a twistor space on $n\mathbb{CP}^2$ and $S \in |F|$ a real irreducible member. Then S is non-singular with $K_S^2 = 8 - 2n$, and the set of twistor lines lying on S is exactly the real part of a real pencil whose self-intersection number is zero. Moreover there is a birational morphism $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ preserving the real structure, such that the twistor lines are mapped to $(1, 0)$ -curves.*

Thus S is always obtained from $\mathbb{CP}^1 \times \mathbb{CP}^1$ by blowing up $2n$ points, where some of the points might be infinitely near in general. As is well-known the position of the blowing up points has a strong effect on algebraic structure of twistor spaces. For example, if a twistor space Z contains S that is obtained from the $2n$ points lying on an irreducible $(1, 2)$ -curve, then it follows $\dim |F| = 2$, and detailed structure of such twistor spaces is investigated by Campana-Kreussler [3]. Then in terms of the configuration of the blowing up points, the genericity of our twistor spaces refers the following property:

Proposition 5.4. *Let Z be a Moishezon twistor space on $4\mathbb{CP}^2$ which is not of Campana-Kreussler type. Suppose that there exists a real irreducible member $S \in |F|$ such that the images of the 8 exceptional curves of the blowing-down $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ in Proposition 5.3 can be taken as distinct points. Then the configuration of the 8 points falls into exactly one of Figure 5.*

For the proof, we first show the following.

Proposition 5.5. *If Z is a twistor space on $4\mathbb{CP}^2$ which satisfies $\dim |F| = 2$, then Z is either a Campana-Kreussler twistor space, or otherwise non-Moishezon.*

Proof. Let $S \in |F|$ be a real irreducible member, which is necessarily non-singular as above. By the assumption, S satisfies $\dim |K_S^{-1}| = 1$. Let $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ be the birational

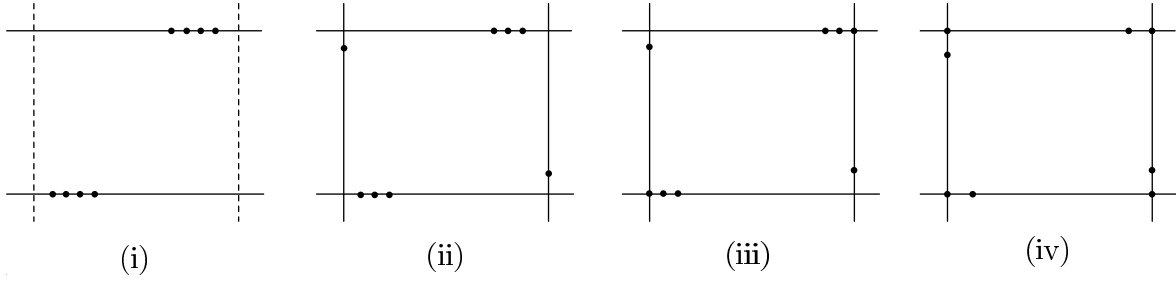


FIGURE 5. Possible configurations of distinct 8 points on $\mathbb{CP}^1 \times \mathbb{CP}^1$ for Moishezon twistor spaces, except for the Campana-Kreussler's case.

morphism fulfilling the properties of Proposition 5.3. (The images of the exceptional curves of ϵ can be infinitely near.) Since ϵ is a composition of blowdowns, by the canonical bundle formula for blowups, the image of the pencil $|K_S^{-1}|$ by ϵ necessarily has to be a pencil of anticanonical curves on $\mathbb{CP}^1 \times \mathbb{CP}^1$; namely a pencil of $(2, 2)$ -curves. Let \mathcal{P} be this pencil on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then again by the canonical bundle formula all the images of the exceptional curves of ϵ must be contained in the base locus of \mathcal{P} .

Suppose that general members of the pencil \mathcal{P} are irreducible. Then the pencil $|K_S^{-1}|$ does not have a fixed component. If this pencil has a base point, by taking a sequence of blowups $\tilde{S} \rightarrow S$, we obtain a morphism $\tilde{S} \rightarrow \mathbb{CP}^1$, which is, again by the canonical bundle formula, necessarily the anticanonical map on \tilde{S} . Therefore, the morphism $\tilde{S} \rightarrow \mathbb{CP}^1$ must be an elliptic fibration. But then by the canonical bundle formula for elliptic surfaces, we obtain $c_1^2(\tilde{S}) = 0$. Since $c_1^2(S) = 0$, this means that \tilde{S} and S are biholomorphic. Hence the pencil $|K_S^{-1}|$ is base point free, and the anticanonical map induces an elliptic fibration $S \rightarrow \mathbb{CP}^1$. This implies the anti-Kodaira dimension of S is one, which means that Z is non-Moishezon.

So in the sequel we suppose that general members of the pencil \mathcal{P} on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are reducible. Then if \mathcal{P} does not have a fixed component, we have $\dim |K_S^{-1}| \geq 2$, which contradicts our assumption. Hence \mathcal{P} has a fixed component. Let C_0 be any one of its irreducible components. Then among the image points of the exceptional curves of ϵ , there exists at least 1 point on C_0 because otherwise C_0 is not a fixed component. Suppose that $C_0 \in |\mathcal{O}(0, 1)|$. Then $\overline{C}_0 \neq C_0$ by the induced real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$, and \overline{C}_0 is also a fixed component of \mathcal{P} . Hence the movable part of \mathcal{P} must be a free 1-dimensional subsystem of $|\mathcal{O}(2, 0)|$, or the system $|\mathcal{O}(1, 0)|$ itself with another fixed $(1, 0)$ -curve C'_0 . But the former cannot occur because general members of the movable part of the 1-dimensional subsystem would be reducible by freeness and both components actually move, so that all the images of the exceptional curves of ϵ have to be contained in $C_0 \cup \overline{C}_0$, which means $\dim |K_S^{-1}| = 2$. So suppose the latter is the case. Then the fixed $(1, 0)$ -curve C'_0 cannot be real, since if so, we would have $C'_0 = \epsilon(L)$ for some twistor line $L \subset S$ by the property of ϵ , whereas on C'_0 there is at least one point among the images of the exceptional curves of ϵ , which means $L^2 < 0$ on S . Hence $\overline{C}'_0 \neq C'_0$. But this is impossible since $C_0 + \overline{C}_0 + C'_0 + \overline{C}'_0$, which is clearly $(2, 2)$ -curves, would be fixed components of the pencil \mathcal{P} of $(2, 2)$ -curves. Thus we obtained $C_0 \notin |\mathcal{O}(0, 1)|$; namely \mathcal{P} does not have a $(0, 1)$ -curve as a fixed component.

Also if a fixed component C_0 is a $(1, 0)$ -curve, then it cannot be real by the same reason. Hence the movable part of \mathcal{P} is a free 1-dimensional subsystem of $|\mathcal{O}(0, 2)|$, or the system $|\mathcal{O}(0, 1)|$ with another fixed component $C'_0 \in |\mathcal{O}(0, 1)|$. But the former implies $\dim |K_S^{-1}| = 2$ by the same argument as above, and the latter cannot occur since this time there is no real $(0, 1)$ -curve. Thus the fixed component C_0 of \mathcal{P} cannot be a $(1, 0)$ -curve. Further we have $C_0 \notin |\mathcal{O}(1, 1)|$, since any $(1, 1)$ -curve is not real and hence $\overline{C}_0 \neq C_0$ has to be also a base curve, which contradicts that \mathcal{P} is a pencil. Similarly we have $C_0 \notin |\mathcal{O}(2, 1)|$ by the real structure. Hence C_0 must be in the remaining possibility, $C_0 \in |\mathcal{O}(1, 2)|$. In this case $\dim |K_S^{-1}| = 1$ means that all the images of the exceptional curves of ϵ belong to C_0 . This implies that the structure of S is exactly as in the case of Campana-Kreussler, and we are done. \square

Proof of Proposition 5.4. First by a result by Kreussler [11, Theorem 6.5], on $n\mathbb{CP}^2$ with $n \geq 3$ we always have $\dim |F| \leq 3$ and the equality holds iff Z is a LeBrun twistor space [13]. Suppose that Z is a LeBrun twistor space. Then it is well-known that a configuration of $2n$ points for generic real irreducible member $S \in |F|$ is as in (i) of Figure 5.

So suppose that Z is a Moishezon twistor space on $4\mathbb{CP}^2$ which is different from LeBrun's nor Campana-Kreussler's, and let S be a real irreducible member of $|F|$ such that the $2n$ points on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are distinct. By Proposition 5.5 we have $\dim |F| = 1$. This means $\dim |K_S^{-1}| = 0$. Let C be the unique anticanonical curve, $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ the birational morphism as in Proposition 5.3 whose images of the exceptional curves are distinct. We put $C_0 := \epsilon(C)$, which is necessarily a real $(2, 2)$ -curve. All the 8 points are on C_0 . If C_0 is irreducible, C_0 must be a non-singular elliptic curve by using the real structure, and from this we readily see that $h^0(mK_S^{-1}) \leq m$ for all $m > 0$, which means that Z is non-Moishezon. Hence C_0 is reducible. Taking the form of the induced real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ into account, we can easily show that the decomposition of C_0 into irreducible components is one of the following 3 types: (a) $(1, 0) + (0, 1) + (1, 0) + (0, 1)$, (b) $(1, 1) + (1, 1)$, both components being irreducible, or (c) $(1, 2) + (1, 0)$, both components being irreducible. We note that since $h^0(K_S^{-1}) = 1$, on any of these components, there exists at least 1 point among the 8 points. Repeating an argument in the last part of the proof of Proposition 5.5, we deduce that (c) cannot happen under our assumption. If C_0 has a multiple component, since there exists no real $(0, 1)$ -curve, it must be a real $(1, 0)$ -curve. But this cannot happen since as remarked above among the 8 points there is at least one point on any irreducible component of C_0 , contradicting the family of twistor lines on S . Hence in both cases (a) and (b) C_0 has no multiple component.

Next we show that in the case (a) there is an irreducible component of C_0 on which precisely 3 points among the 8 points lie. If not, then because we are excluding LeBrun twistor spaces, on each of the 4 irreducible components exactly 2 points are lying among the 8 points. In this case, the restriction $K_S^{-1}|_C \simeq [C]|_C$ belongs to $\text{Pic}^0 C \simeq \mathbb{C}^*$, which again implies $h^0(-mK_S) \leq m$ for any $m > 0$ as in the above case. This implies that Z is not Moishezon. Hence the component actually exists.

Next we prove that if C_0 is in the case (b), there exists another birational morphism $\epsilon' : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ preserving the real structure such that $C'_0 = \epsilon'(C)$ falls into the case (a), and such that the images of twistor lines in S are $(1, 0)$ -curves. For this we write $C_0 = C_1 + \overline{C}_1$ with C_1 and \overline{C}_1 being irreducible $(1, 1)$ -curves. If exactly 4 points belong to C_1 , then the remaining 4 points belong to \overline{C}_1 , and also no point coincides with the 2 points $C_1 \cap \overline{C}_1$. Then by a similar reason for the case (a), this implies that Z is not

Moishezon. So the 2 points $C_1 \cap \overline{C}_1$ are included in the 8 points. Therefore ϵ factors as $S \rightarrow S_1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ where the latter arrow is the blowup at $C_1 \cap \overline{C}_1$. We obtain 6 points on S_1 as the images of the exceptional curves of $S \rightarrow S_1$. These 6 points are not on the exceptional curves of $S_1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ by the assumption that the 8 points are distinct. This implies that the unique anticanonical curve C on S is a cycle of 4 rational curves, whose self-intersection numbers are $(-3), (-1), (-3), (-1)$. Then for another blowdown ϵ' in the proposition, it is enough to choose a blowingdown $S_1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ which does not contract the 2 exceptional curves of the original $S_1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$, and composite it with the morphism $S \rightarrow S_1$. Thus we obtained another ϵ' as claimed, and we can neglect the case (b).

Hence C_0 can be supposed to be in the case (a) and that there is at least one component on which exactly 3 points among 8 points lie. This directly means the 8 points have to be put on C_0 arranged as in (ii), (iii) or (iv), as claimed. \square

Needless to say, the case (ii) of Proposition 5.4 is exactly the situation we have investigated in this paper. Since it is clear that all other 3 cases ((i), (iii) and (iv)) can be obtained as small deformations of the case (ii), it would be reasonable to say that among the surface S obtained from the 8 points arranged as in (i)–(iv), the case (ii) is most generic. By deformation theory including a co-stability theorem of Horikawa [9], the same is true for the twistor spaces containing these surfaces. Namely any twistor spaces on $4\mathbb{CP}^2$ which has S obtained from (i), (iii) and (iv) as a real member of $|F|$ can be obtained as a limit of the twistor spaces investigated in this paper. In particular, the present twistor spaces can be obtained as a small deformation of a LeBrun twistor space, and this proves the existence of our twistor spaces. We also remark that by using the Horikawa's theorem, it is possible to show that the present twistor spaces can also be obtained as a small deformation of the twistor spaces studied in [8] (on $4\mathbb{CP}^2$, of course). These are the reason why we call the present twistor spaces to be generic.

Finally we remark that a converse of Proposition 5.4 also holds. Namely if a twistor space Z on $4\mathbb{CP}^2$ has real irreducible $S \in |F|$ which is obtained from the 8 points in the case (ii), (iii), or (iv), then $\dim |F| = 1$ and Z is Moishezon. Concerning structure of these twistor spaces, the case (iii) can be regarded as a mild degeneration of the present twistor spaces, in the sense that the twistor space still has a double covering structure over the scroll Y by the anticanonical system. These twistor spaces (having S obtained from the configuration (iii)) are analogous to a double solid twistor spaces on $3\mathbb{CP}^2$ of a degenerate form found by Kreussler-Kurke [12, p. 50, Case (b)]. Here we mention that there is *one more* another type of twistor spaces on $4\mathbb{CP}^2$ of a degenerate form which do not admit \mathbb{C}^* -action. We will study these 2 kinds of double solid twistor spaces on $4\mathbb{CP}^2$ in a separate paper. On the other hand, although Moishezon, it turns out that the remaining case (iv) does not have structure of double solids, because the anticanonical map of the twistor spaces becomes birational. So they are rather similar to the twistor spaces of Joyce metrics on $4\mathbb{CP}^2$ of non-LeBrun type [7]. But contrary to the Joyce's case, explicit realization of the anticanonical model seems difficult.

6. APPENDIX: INVERTING THE CONTRACTION MAP $Z_3 \rightarrow Z_4$ BY A BLOWUP

We recall from Section 3.2 that the singular variety Z_4 is obtained from the twistor space Z via non-singular spaces Z_1, Z_2 and Z_3 , and the transformations therein are standard until getting Z_3 . On the other hand, the map $\mu_4 : Z_3 \rightarrow Z_4$ contracts the reducible connected

divisor $E_1 \cup E_1$ to a reducible connected curve $l_4 \cup \bar{l}_4$, and there we used Fujiki's contraction theorem. One would wish to find such a birational morphism through the usual procedure of blowups with non-singular center. In this subsection, we explicitly see that an embedded blowup at and a small resolution provide the desired situation and point out that the process is in a sense a singular version of the Hironaka's construction of non-projective Moishezon 3-folds.

For this we consider the double covering of Y with branch B which is the intersection with the quartic hypersurface (4.11). Recall that $l = \{z_0 = z_1 = z_2 = 0\}$, and $B \cap l$ consists of two points $\{Q = 0\} \cap l$. As before let p_2 be any one of the two points and we work in a neighborhood of p_2 as the situation around \bar{p}_2 can see by just taking the image under the real structure. As in the proof of Proposition 4.8, putting $x = z_0/z_4, y = z_1/z_4, z = z_2/z_4$ and $u = Q$, we can use (x, y, z, u) as coordinates in a neighborhood of p_2 in \mathbb{CP}^4 , and we may suppose that the hyperquartic (4.11) is defined by $x = u^2$, while Y is defined by $x^2 = yz$. Next for studying the structure of the double covering, we introduce another coordinate w over the neighborhood of p_2 , so that the double cover is defined by

$$(6.1) \quad x^2 = yz, \quad w^2 = u^2 - x \text{ in } \mathbb{C}^5 \text{ with coordinates } (x, y, z, u, w).$$

(This is the equation of Z_4 around the points $l_4 \cap \bar{l}_4$ we promised in Section 3.2.) Let W be this double covering and $\varpi : W \rightarrow Y$ the projection. (Of course this is valid only in a neighborhood of p_2 .) The singular locus of W is

$$(6.2) \quad \varpi^{-1}(l) = \{x = y = z = u - w = 0\} \cup \{x = y = z = u + w = 0\},$$

which is a union of 2 lines, and W has A_1 -singularities along these lines minus the origin. We note that substituting $x = u^2 - w^2$ to $x^2 = yz$, we obtain that W contains the following distinguished 4 surfaces

$$(6.3) \quad \{x = y = w - u = 0\}, \quad \{x = z = w - u = 0\},$$

$$(6.4) \quad \{x = y = w + u = 0\}, \quad \{x = z = w + u = 0\}.$$

Then obviously we have $W \cap \{x = y = z = 0\} = \varpi^{-1}(l)$. So if we let $\tilde{\mathbb{C}}^5 \rightarrow \mathbb{C}^5$ to be the blowup at the plane $\{x = y = z = 0\}$ and \tilde{W} to mean the strict transform of W , then $\tilde{W} \rightarrow W$ is an embedded blowup at $\text{Sing } W$. Then by concrete computations using coordinates it is not difficult to see that the exceptional locus of $\tilde{W} \rightarrow W$ consists of 2 irreducible divisors which are over the 2 lines (6.2) respectively, that the inverse image of the origin is a non-singular rational curve, and that the singularities of \tilde{W} consists of 2 points lying on this rational curve, both of which are ordinary double points. Further, by the effect of the blowup, the pair of planes (6.3), both of which contain the same line $\{x = y = z = w - u = 0\}$ are separated by one of the exceptional divisors, and the same for another pair (6.4). This way we get the situation of Figure 3, (e). Then an appropriate small resolution (which is obvious from the figure) gives the desired space Z_3 .

As above the center of the blowup is a reducible curve whose fundamental group is \mathbb{Z} . Thus, together with an inspection of the choice of the small resolution displayed in Figure (3), it would be possible to say that the transformation from Z_4 to Z_3 is a singular version of Hironaka's well-known example of non-projective Moishezon 3-folds [5] in the sense that the center of the blowup in the present situation is a singular locus of the 3-fold.

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